ADAPTATION WITH CONSTANT GAINS: ANALYSIS FOR FAST VARIATIONS

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Abstract: Adaptation laws with constant gains, that adjust parameters of linear regression models, are investigated. The class of algorithms includes LMS as its simplest member, while other algorithms such as Wiener LMS may improve performance by including linear filters. Expressions in closed form for the tracking MSE are obtained for rapidly varying parameters of FIR systems with white inputs. This situation may occur in e.g. the tracking of fading communication channels. Stability and convergence in MSE are ascertained by the stability of a transfer function, without assuming independent regressor vectors. A key technique for obtaining these results is a transformation of these adaptation algorithms into linear time-invariant filters, called learning filters, that operate in open loop for slow parameter variations.

1. INTRODUCTION

Consider discrete-time and possibly complex-valued measurements generated by a linear regression

$$y_t = \varphi_t^* h_t + v_t \quad , \tag{1}$$

where y_t is a scalar measured signal, v_t is a noise while φ_t^* is a regression vector, which is known at discrete time t. The parameter vector h_t, with known dimension M $\times$ 1, is to be estimated.

We shall here investigate a class of linear time-invariant estimators

$$\hat{h}_{t+k|t} = \boldsymbol{\mathcal{M}}_k(q^{-1})(\varphi_t \varepsilon_t) \quad , \tag{2}$$

where $\boldsymbol{\mathcal{M}}_k(q^{-1})$ represents a causal rational matrix in the the backward shift operator q^{-1} ($q^{-1}y_t = y_{t-1}$) and where

$$\varphi_t \varepsilon_t = \varphi_t(y_t - \varphi_t^* \hat{h}_{t|t-1}) \tag{3}$$

is the negative gradient of $|\varepsilon_t|^2/2 = |y_t - \hat{y}_{t|t-1}|^2/2$, with $\hat{y}_{t|t-1} = \varphi_t^* \hat{h}_{t|t-1}$ being the one-step prediction of y_t.

Above, $\hat{h}_{t+k|t}$ is an estimate of h_{t+k} obtained at time t by filtering ($k = 0$), prediction ($k > 0$) or fixed lag smoothing ($k < 0$). Most known linear algorithms with constant gains, related to the multi-step algorithms discussed by Benveniste [1], fit into this structure [2, 4]. For the LMS algorithm $\hat{h}_{t+1|t} = \hat{h}_{t|t-1} + \mu_{\text{LMS}} \varphi_t \varepsilon_t$,

$$\boldsymbol{\mathcal{M}}_1(q^{-1}) = \frac{\mu_{\text{LMS}}}{1 - q^{-1}} \mathbf{I} \quad . \tag{4}$$

Wiener methods for adjusting $\boldsymbol{\mathcal{M}}_k(q^{-1})$ have been developed in [2],[4],[5]. In this presentation, we outline an analysis of adaptation laws (2),(3) in which $\boldsymbol{\mathcal{M}}_k(q^{-1})$ is confined to be an arbitrary diagonal transfer function matrix. Stability and the resulting estimation error

$$\tilde{h}_{t+k|t} = h_{t+k} - \hat{h}_{t+k|t}$$

is investigated for FIR systems with time-varying parameters. Closed-form expressions are presented for the trace of the steady-state parameter tracking error covariance matrix

$$\mathbf{P}_k \triangleq \lim_{t\to\infty} \mathbf{P}_{t+k|t} = \lim_{t\to\infty} \mathrm{E}\,(\tilde{h}_{t+k|t} \tilde{h}_{t+k|t}^*) \quad . \tag{5}$$

We make the following basic assumptions.

Assumption 1: The noise v_t is white, stationary and zero mean with variance σ_v^2, while φ_t^* is stationary with zero mean and nonsingular covariance matrix $\mathbf{R} = \mathrm{E}\,(\varphi_t \varphi_t^*)$. Moreover, h_t, v_t, and φ_t^* are mutually independent □

2. THE LEARNING FILTER

The algorithm (2),(3) will now be expressed as a causal filter, denoted the *learning filter* $\boldsymbol{\mathcal{L}}_k(q^{-1})$, that operates on a signal vector called the fictitious measurement

$$f_t \;\triangleq\; \varphi_t \varepsilon_t + \mathbf{R} \hat{h}_{t|t-1} \quad . \tag{6}$$

Since $\hat{h}_{t|t-1} = q^{-1} \boldsymbol{\mathcal{M}}_1(q^{-1})(\varphi_t \varepsilon_t)$, (6) gives

$$\varphi_t \varepsilon_t = (\mathbf{I} + q^{-1} \mathbf{R} \boldsymbol{\mathcal{M}}_1(q^{-1}))^{-1} f_t \quad .$$

The estimator (2) can thus be expressed as

$$\begin{aligned} \hat{h}_{t+k|t} &= \boldsymbol{\mathcal{M}}_k(q^{-1})(\mathbf{I} + q^{-1} \mathbf{R} \boldsymbol{\mathcal{M}}_1(q^{-1}))^{-1} f_t \\ &\triangleq \boldsymbol{\mathcal{L}}_k(q^{-1}) f_t \;=\; \sum_{i=0}^{\infty} \mathbf{L}_i^k f_{t-i} \quad . \end{aligned} \tag{7}$$

By (3) and (1),

$$\varphi_t \varepsilon_t = \varphi_t \varphi_t^* \tilde{h}_{t|t-1} + \varphi_t v_t \quad . \tag{8}$$

Adding and subtracting $\mathbf{R}\tilde{h}_{t|t-1}$ in (8) gives

$$\varphi_t \varepsilon_t = \mathbf{R} h_t - \mathbf{R} \hat{h}_{t|t-1} + (\varphi_t \varphi_t^* - \mathbf{R}) \tilde{h}_{t|t-1} + \varphi_t v_t \quad . \tag{9}$$

Define

$$\mathbf{Z}_t \;\stackrel{\Delta}{=}\; \varphi_t\varphi_t^* - \mathbf{R} \tag{10}$$

$$\eta_t \;\stackrel{\Delta}{=}\; \mathbf{Z}_t\tilde{h}_{t|t-1} + \varphi_t v_t \tag{11}$$

and denote $\mathbf{Z}_t$ and η_t the *autocorrelation matrix noise* and the *gradient noise*, respectively. By inserting (9) into (6), we now obtain

$$f_t = \mathbf{R}h_t + \mathbf{Z}_t\tilde{h}_{t|t-1} + \varphi_t v_t = \mathbf{R}h_t + \eta_t \ . \tag{12}$$

The point of this transformation is that we have decomposed the feedback inherent in the algorithm into two parts: An inner time-invariant loop via $\mathbf{R}$, which is absorbed into the definition of $\mathcal{L}_k(q^{-1})$ in (7); and an outer time-varying feedback via $\mathbf{Z}_t\tilde{h}_{t|t-1}$, see Fig. 1, which we call the *feedback noise*.

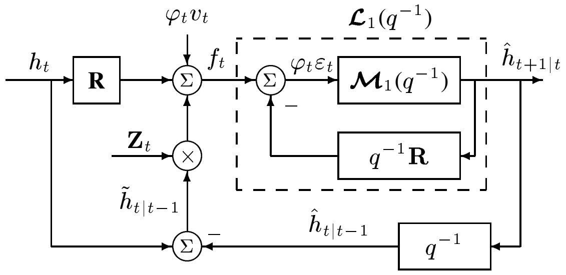

Fig. 1. The learning filter operates on f_t in which h_t is the signal, while η_t plays the role of noise. The feedback loop around $\mathcal{L}_1(q^{-1})$ via the feedback noise $\mathbf{Z}_t\tilde{h}_{t|t-1}$ can be neglected in the case of slow variations. For fast variations, it may cause instability.

A necessary, but not sufficient, condition for stability is *internal stability of the learning filter* (representing the inner loop). Optimal Wiener design, presented in [2], [5], results in stable learning filters $\mathcal{L}_k(q^{-1})$. For adaptation laws obtained by other means, stability must be verified separately.

In general, the outer feedback via the feedback noise also has to be taken into account. A sufficient but conservative condition for stability of this feedback is for bounded regressors provided by the small gain theorem [6]: For causal and L_p-stable $\mathcal{L}_1(q^{-1})\mathbf{Z}_t$, stability is preserved if

$$||q^{-1}\mathcal{L}_1(q^{-1})\mathbf{Z}_t\tilde{h}_{t|t-1}||_p \leq \gamma||\tilde{h}_{t|t-1}||_p \ \ ; \ \ \gamma < 1 \ .$$

Our aim here will be to present a less conservative stability condition and also present performance estimates for the case of FIR systems with white input data.

3. THE TRACKING MSE

By (7) and (12), the tracking error can be expressed as

$$\tilde{h}_{t+k|t} = (\mathbf{I} - q^{-k}\mathcal{L}_k\mathbf{R})h_{t+k} - \mathcal{L}_k\varphi_t v_t - \mathcal{L}_k\mathbf{Z}_t\tilde{h}_{t|t-1} \ . \tag{13}$$

Thus, three terms affect the tracking error: The *lag error* $(\mathbf{I} - q^{-k}\mathcal{L}_k\mathbf{R})h_{t+k}$, an error term $\mathcal{L}_k\varphi_t v_t$ caused by measurement noise, and a feedback noise term $\mathcal{L}_k\mathbf{Z}_t\tilde{h}_{t|t-1}$ influenced by old parameter tracking errors.

By Assumption 1, h_t and $\varphi_t v_t$ are independent. If $\tilde{h}_{t+k|t}$ is stationary, then the error covariance matrix (5) becomes

$$\mathbf{P}_k = \mathbf{V}_h^k + \mathbf{V}_{\varphi v}^k + \mathbf{V}_{Z\tilde{h}}^k + \mathbf{V}_{hZ\tilde{h}}^k + \mathbf{V}_{\varphi vZ\tilde{h}}^k \tag{14}$$

where

$$\mathbf{V}_h^k \;=\; \mathrm{E}\,(\mathbf{I} - q^{-k}\mathcal{L}_k\mathbf{R})h_{t+k}\left((\mathbf{I} - q^{-k}\mathcal{L}_k\mathbf{R})h_{t+k}\right)^* \tag{15}$$

$$\mathbf{V}_{\varphi v}^k \;=\; \mathrm{E}\,(\mathcal{L}_k\varphi_t v_t)(\mathcal{L}_k\varphi_t v_t)^* \tag{16}$$

$$\mathbf{V}_{Z\tilde{h}}^k \;=\; \mathrm{E}\,(\mathcal{L}_k\mathbf{Z}_t\tilde{h}_{t|t-1})(\mathcal{L}_k\mathbf{Z}_t\tilde{h}_{t|t-1})^* \ . \tag{17}$$

The last two terms in (14) are due to correlations between the feedback noise, the lag error and $\varphi_t v_t$, respectively:

$$\begin{aligned}
\mathbf{V}_{hZ\tilde{h}}^k &= -\mathrm{E}\,\{(\mathbf{I} - q^{-k}\mathcal{L}_k\mathbf{R})h_{t+k}(\mathcal{L}_k(\mathbf{Z}_t\tilde{h}_{t|t-1}))^*\} \\
&\quad - \mathrm{E}\,\{((\mathbf{I} - q^{-k}\mathcal{L}_k\mathbf{R})h_{t+k})^*\mathcal{L}_k(\mathbf{Z}_t\tilde{h}_{t|t-1})\} \\
\mathbf{V}_{\varphi vZ\tilde{h}}^k &= \mathrm{E}\,\{\mathcal{L}_k(\varphi_t v_t)(\mathcal{L}_k(\mathbf{Z}_t\tilde{h}_{t|t-1}))^*\} \\
&\quad + \mathrm{E}\,\{(\mathcal{L}_k(\varphi_t v_t))^*\,\mathcal{L}_k(\mathbf{Z}_t\tilde{h}_{t|t-1})\} \ .
\end{aligned}$$

From a tracking point of view, the elements of h_t can be regarded as slowly time-varying if the feedback noise can be neglected when calculating the tracking performance [7]. This definition is closely related to other definitions of the property of slow variations [8, 9]. The tracking performance can then be obtained by evaluating only the first two terms in (14). This is illustrated in [7] for algorithms with stable learning filters of arbitrary structure.

Non-negligible feedback noise can be handled whenever successive regression vectors φ_t^* can be assumed independent, see e.g. [10, 11, 12] for LMS results based on this independence assumption. For $\mathcal{L}_k$ of general structure, closed-form expressions can then be obtained for term 3 in (14), $\mathbf{V}_{Z\tilde{h}}^k$, while the two last terms vanish [3]. Unfortunately, regressor vector independence is a restrictive assumption, which is not valid when the regressor elements consist of lagged outputs from a dynamic system.

However, for FIR systems with complex-valued white inputs, we can obtain good approximations for $\mathbf{V}_{Z\tilde{h}}^k$, *without* assuming independent regression vectors.

4. TRACKING OF FAST VARIATIONS IN FIR SYSTEMS WITH WHITE INPUTS

Consider the scalar finite impulse response system (1) with

$$\varphi_t^* = (\ u_t\ u_{t-1} \ldots u_{t-\mathrm{M}+1}\ ) \ , \tag{18}$$

and assume the input data u_t to be white with zero mean and variance σ_u^2 ($\mathbf{R} = \sigma_u^2 \mathbf{I}$).

Closed form expressions are now derived for $\mathrm{tr}\,(\mathbf{P}_k)$ under the following approximations:

Approximation 1:

$$\mathrm{tr}\,\mathrm{E}\,(\mathbf{Z}_\tau^* \mathbf{Z}_t \tilde{h}_{t|t-1} \tilde{h}_{\tau|\tau-1}^*) = \mathrm{tr}\,\mathrm{E}\,(\mathbf{Z}_\tau^* \mathbf{Z}_t)\,\mathrm{E}\,(\tilde{h}_{t|t-1} \tilde{h}_{\tau|\tau-1}^*) \tag{19}$$

Approximation 2: The feedback noise $\mathbf{Z}_t \tilde{h}_{t|t-1}$ is uncorrelated with $\varphi_\tau v_\tau$ and $h_\tau, \forall \tau$.

Approximations 1 and 2 can be shown to hold exactly for $\mathrm{M} = 2$ when input data has constant modulus [3]. They do not hold exactly for $\mathrm{M} > 2$, but turn out to be good approximations for FIR systems of higher order.

Independence between $\mathbf{Z}_t$ and $\tilde{h}_{t|t-1}$ would *imply* (19), but such an assumption is not *equivalent* to (19); it would place unnecessarily strong restrictions on the statistics.

Under Approximation 2, the two last cross-terms in (14) are neglected. This is not necessary, but it simplifies the calculations.

The family of tracking algoritms is now confined to a "Wiener LMS" algorithm structure [4] that for white regressors utilizes diagonal stable learning filters

$$\boldsymbol{\mathcal{L}}_k(q^{-1}) \;=\; \frac{Q_k(q^{-1})}{\beta(q^{-1})} \frac{1}{\sigma_u^2} \mathbf{I} \;=\; \sum_{i=0}^{\infty} \mathrm{L}_i^k \,\mathbf{I}\, q^{-i} \quad . \tag{20}$$

Here, $Q_1(q^{-1})$ and $\beta(q^{-1})$ are polynomials[1]. For LMS (4), (7) gives

$$\boldsymbol{\mathcal{L}}_1(q^{-1}) = \frac{\mu_{\text{LMS}}}{1 - (1 - \mu_{\text{LMS}} \sigma_u^2) q^{-1}} \mathbf{I} \quad . \tag{21}$$

By (7), (12) and (20), the tracking error becomes

$$\begin{aligned} \tilde{h}_{t+k|t} &= (\mathbf{I} - q^{-k} \boldsymbol{\mathcal{L}}_k(q^{-1}) \sigma_u^2) h_{t+k} - \boldsymbol{\mathcal{L}}_k(q^{-1}) \eta_t \quad (22) \\ &= \frac{\beta(q^{-1}) - q^{-k} Q_k(q^{-1})}{\beta(q^{-1})} h_{t+k} - \frac{Q_k(q^{-1})}{\sigma_u^2 \beta(q^{-1})} \eta_t \quad . \end{aligned}$$

Introducing the Pearson kurtosis[2], defined as

$$\kappa_u \triangleq \mathrm{E}\,|u_t|^4 / (\mathrm{E}\,|u_t|^2)^2 \quad , \tag{23}$$

the main result can now be presented.

Theorem 1: Under Assumption 1, consider the FIR system (1) (18) with white input data u_t that are either circular complex for arbitrary FIR degree M or real-valued, for $\mathrm{M} \leq 2$. Let the parameter vector h_t be estimated by (6)-(7) with a stable $\boldsymbol{\mathcal{L}}_k(q^{-1})$ having the structure (20), resulting in stationary lag errors with finite second order moments. Under Approximation 1 and 2, a finite steady state mean square parameter error $\mathrm{tr}\,(\mathbf{P}_k)$ then exists if and only if

$$\mathcal{G}(z^{-1}) \;=\; \frac{1}{1 - m\sigma_u^4 \sum_{i=0}^{\infty} |\mathrm{L}_i^1|^2 z^{-i-1}} \tag{24}$$

is stable, where

$$m \;\triangleq\; \kappa_u + \mathrm{M} - 2 \quad . \tag{25}$$

The k-step estimation error is given by

$$\mathrm{tr}\,(\mathbf{P}_k) = \mathrm{tr}\,(\mathbf{V}_h^k) + \mathrm{tr}\,(\mathbf{V}_{\varphi v}^k) + \mathrm{tr}\,(\mathbf{V}_{Z\tilde{h}}^k) \tag{26}$$

where

$$\begin{aligned} \mathrm{tr}\,(\mathbf{V}_h^k) &= \Big\|\frac{\beta(q^{-1}) - q^{-k} Q_k(q^{-1})}{\beta(q^{-1})} h_{t+k}\Big\|_2^2 \quad (27) \\ \mathrm{tr}\,(\mathbf{V}_{\varphi v}^k) &= \mathrm{M}\frac{\sigma_v^2}{\sigma_u^2} \Sigma_k \quad (28) \\ \mathrm{tr}\,(\mathbf{V}_{Z\tilde{h}}^k) &= m\,\mathrm{tr}\,(\mathbf{P}_1) \Sigma_k \quad (29) \end{aligned}$$

in which

$$\Sigma_k \;\triangleq\; \frac{1}{2\pi j} \oint_{|z|=1} \left| \frac{Q_k(z^{-1})}{\beta(z^{-1})} \right|^2 \frac{dz}{z} \quad , \tag{30}$$

$$\mathrm{tr}\,(\mathbf{P}_1) = \frac{\mathrm{tr}\,(\mathbf{V}_h^1) + \mathrm{M}\frac{\sigma_v^2}{\sigma_u^2} \Sigma_1}{1 - m\Sigma_1} \quad . \tag{31}$$

Proof: See [3] □

Note the FIR order M in (25). The allowed gain of $\boldsymbol{\mathcal{L}}_1(q^{-1})$, determined by (24), therefore decreases with M. Also, note the dependence on the kurtosis κ_u.

The lag error term (27) can be calculated if we assume the second order moments (spectral density) of h_t to be given. Note, however, that if drifting parameters (random walks) are assumed, then the adaptation law must be designed so that the lag error covariance (27) remains finite. Polynomial Wiener designs [4] have this property.

A separate expression (31) is provided for the tracking error of one-step prediction estimates $\mathrm{tr}\,(\mathbf{P}_1)$. This factor enters the general expression (29) for the feedback noise contribution, due to the feedback via $\hat{h}_{t|t-1}$ in Fig. 1.

5. EXAMPLE

The validity of Approximations 1 and 2 and the accuracy of the results presented in Theorem 1 will be investigated next. Consider the tracking of a time-varying FIR system (1)(18) with white complex-valued regressors, $u_t \in \{1, -1, i, -i\}$, with all four values having equal probability ($\kappa_u = 1$). The parameter dynamics is governed by the oscillatory system

$$h_t = 2p \cos \omega_o h_{t-1} - p^2 h_{t-2} + e_t \quad , \tag{32}$$

[1] With $\boldsymbol{\mathcal{L}}_k$ and $\mathbf{R}$ being diagonal, the filter $\boldsymbol{\mathcal{M}}_k$ in (2), obtained via (7), will also be diagonal.

[2] For constant modulus data, $\kappa_u = 1$, for circular complex-valued Gaussian data $\kappa_u = 2$, and for real-valued Gaussian regressors, $\kappa_u = 3$.

here with $\omega_o = 0.05$ and $p = 0.995$. We select the additive noise variance $\sigma_v^2 = 0.01$ and $\mathrm{E}\,(e_t e_t^*) = (\lambda_e/\mathrm{M})\mathbf{I}$, where λ_e is tuned to give $\mathrm{E}\,|h_t|^2 = 1$, resulting in an output SNR of 20 dB. This corresponds to a case with fast variations, where the feedback noise is significant. One-step prediction estimates are obtained by LMS (21), with a gain μ_{LMS} tuned to minimize $\mathrm{tr}\,(\mathbf{P}_1)$ in (31). Table 1 displays the five contributing terms to the minimal criterion value (14), obtained from Theorem 1 (bold figures). The feedback noise-related terms are also estimated by simulation over 100000 data (italic figures).

The two cross-terms, which are neglected in Theorem 1 by Approximation 2, are seen to be small as compared to $\mathrm{tr}\,(\mathbf{V}^1_{Z\tilde{h}})$. The relative difference between right- and left-hand terms in (19), called the error in (19) in Table 1, is small (around 12% or less).

The optimal LMS adaptation gain μ^{opt}_{LMS} is accurately predicted by the theory. (For larger M, $\mathrm{M}\mu^{opt}_{\text{LMS}} \to 0$ since adaptation will then become ineffective. Therefore, $\min \mathrm{tr}\,(\mathbf{P}_1) \to \mathrm{E}\,|h_t|^2 = 1$ when $\mathrm{M} \to \infty$.)

The LMS performance is finally compared with a simplified Wiener LMS (SWLMS) tracker, derived in [4], based on the model (32). In (20), $Q_1(q^{-1})$ and $\beta(q^{-1})$ are then

$$Q_1(q^{-1}) = \mu\left(\frac{-d_1}{1 + d_2(1-\mu)} - d_2 q^{-1}\right) \qquad (33)$$

$$\beta(q^{-1}) = 1 + \frac{1 + d_1(1+d_2)(1-\mu)}{1 + d_2(1-\mu)} q^{-1} + d_2(1-\mu)q^{-2} \quad , \qquad (34)$$

where d_1 and d_2 are obtained for second order AR statistics (32) as

$$d_1 = -2p\cos\omega_o \quad , \quad d_2 = p^2 \ . \qquad (35)$$

As for the LMS algorithm, a scalar step-size parameter μ is tuned to minimize $\mathrm{tr}\,(\mathbf{P}_1)$ in (31). Comparing the last two lines of Table 1 (tracking MSE for SWLMS) to the first two lines (LMS), it is evident that SWLMS can provide superior performance, when using filtering that is matched to the dynamics of h_t. Robust performance for mismatched designs can also be investigated by Theorem 1, see [13].

Table 1. Contributions to the asymptotic tracking error $\mathrm{tr}\,\mathbf{P}_1$ when FIR models of order M are tracked by LMS and Simplified Wiener LMS (SWLMS). Theoretical predictions from Theorem 1 (bold) are compared to simulation results (in italics).

FIR order	M:	2	4	10	20
LMS:	$\mathrm{tr}\,(\mathbf{P}_1)$	**.0249**	**.0721**	**.3490**	**.9249**
		.0250	*.0644*	*.3008*	*.8830*
Lag error:	$\mathrm{tr}\,(\mathbf{V}^1_h)$	**.0099**	**.0268**	**.1311**	**.4505**
Noise:	$\mathrm{tr}\,(\mathbf{V}^1_{\varphi v})$	**.0067**	**.0071**	**.0067**	**.0053**
Feedback:	$\mathrm{tr}\,(\mathbf{V}^1_{Z\tilde{h}})$	**.0083**	**.0381**	**.2112**	**.4691**
		.0084	*.0335*	*.1827*	*.4309*
	$-\mathrm{tr}\,(\mathbf{V}^1_{hZ\tilde{h}})$	*0*	*.0038*	*.0235*	*.0278*
	$\mathrm{tr}\,(\mathbf{V}^1_{\varphi vZ\tilde{h}})$	*0*	*.0002*	*.0007*	*.0005*
Error in	(19):	0	*6.2%*	*11.3%*	*12.2%*
Optimal	$\mathrm{M}\,\mu^{opt}_{\text{LMS}}$	**1.00**	**1.18**	**1.24**	**1.00**
gain:		*1.00*	*1.20*	*1.26*	*1.04*
SWLMS:	$\mathrm{tr}\,(\mathbf{P}_1)$	**.0090**	**.0218**	**.0893**	**.2249**
		.0091	*.0199*	*.0774*	*.2063*

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