A NOVEL DERIVATION METHODOLOGY FOR POLYNOMIAL-LQ CONTROLLER DESIGN

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Abstract

A simple derivation methodology for optimization of linear quadratic controllers is presented, in the discrete time polynomial systems framework. A control law variation, regarded as a potential feedforward from the innovations, is used. Orthogonality is evaluated in the frequency domain, by collectively cancelling unstable poles by zeros. The suggested method, summarized as a three–step scheme, is exemplified on a disturbance measurement feedforward and an output feedback problem. It is a simple and more direct alternative to the "completing the squares" approach.

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1 INTRODUCTION

The study of linear quadratic control laws in input–output form has been a subject of long–standing interest. The Wiener solution of Youla et al [2] and use of the polynomial approach of Kučera [3],[4] are two well–known contributions. For LQ problems, the polynomial equations approach provides a systematic way of evaluating the causal factors of a classical Wiener–Hopf solution. Efficient numerical algorithms for solving polynomial equations exist [3],[6]. For a given LQ problem, the corresponding polynomial equations can be derived, following Kučera, by completing the squares in the criterion. See, for example, [3], [4], [9], [15], [18].

We have found the method of completing the squares to be rather cumbersome. A new way of deriving the design equations, which requires significantly fewer algebraic steps, is suggested here. Essentially, an old variational argument is utilized in a novel way. Orthogonality between signals and variations is evaluated in the frequency domain, to obtain polynomial equations which define the control law. The method is a short, simple and direct alternative for use within the polynomial system framework. When it is used, the resulting equations are, of course, the same that would have been obtained by a "completing the squares"–reasoning. Subsequent discussion of solvability of the equations and stability of the solution remains unaltered. The use of the method in discrete–time control problems will be explained. Application to continuous–time problems is straightforward. In a companion paper [1], based on [8], the optimization of estimators is considered.

An outline of the technique is presented in Section II. Application to open–loop problems is considered in Section III, exemplified in detail by a disturbance measurement feedforward problem. Feedback control is discussed in Section IV, where an output feedback problem is considered. The reasoning is summarized as a step–by–step procedure in Section V.

Remark on the notation

The backward shift operator is denoted q^{-1}. It corresponds to z^{-1} in the frequency domain. Trace and transpose of matrices M are denoted tr$M$ and M', respectively. For any polynomial matrix $P(z^{-1})$, $P_* = P(z)'$. The arguments are often omitted. A square polynomial matrix, of full normal rank, is called *stable* (or strictly Schur), if its determinant has all zeros in $|z| < 1$. Rational matrices $\mathcal{R}(z^{-1})$ are called stable if all their elements are transfer functions with poles in $|z| < 1$. If $P(q^{-1})$ is a square polynomial matrix, all elements of the rational matrix $P(q^{-1})^{-1}$ are *causal* if and only if the leading coefficient matrix of $P(q^{-1})$, denoted $P(0)$, is nonsingular. The *degree* of $P(q^{-1})$ is the highest degree of any of its polynomial elements.

2 OUTLINE OF THE PROCEDURE

Consider the control of a linear discrete–time system, which is stochastic and time–invariant. Its inputs $u(t) \in R^m$ are to be calculated, based on linear combinations of measurable outputs $z(t) \in R^n$, so that the signals $y(t) \in R^p$ are controlled. Denote the regulator

$$u(t) = -\mathcal{R}(q^{-1})z(t) \tag{2.1}$$

where $\mathcal{R}(z^{-1})$ is a causal rational $m|n$–matrix, to be designed so that the controlled system is stable and the infinite–horizon quadratic criterion

$$J = E\{\mathrm{tr}Vy(t)(Vy(t))' + \mathrm{tr}Wu(t)(Wu(t))'\} \tag{2.2}$$

is minimized. Above, $V(q^{-1})$ and $W(q^{-1})$ are polynomial weighting matrices, of dimensions $p|p$ and $m|m$, respectively. (Use of rational weighting matrices is straightforward, but requires additional coprime factorizations.) Variational arguments will be used in order to minimize (2.2). For this purpose, introduce the *alternative regulator*

$$u(t) = -\mathcal{R}(q^{-1})z(t) + n(t) \tag{2.3}$$

where $n(t) \in R^m$ is a linear function of measurements up to time t. The only restriction on this variational term is that the control law is causal and the total system is internally stable. The use of (2.3) results in the modified signals

$$\begin{aligned} y(t) &= y_o(t) + \delta y(t) \\ u(t) &= u_o(t) + \delta u(t) \end{aligned} \tag{2.4}$$

where $y_o(t)$ and $u_o(t)$ result from control by (2.1), while $\delta y(t)$ and $\delta u(t)$ are caused by the variation $n(t)$. The criterion can then be expressed as

$$J = J_o + 2J_1 + J_2 \tag{2.5}$$

where

$$\begin{aligned} J_o &= E(\mathrm{tr}(Vy_o)(Vy_o)' + \mathrm{tr}(Wu_o)(Wu_o)') \\ J_1 &= E(\mathrm{tr}(Vy_o)(V\delta y)' + \mathrm{tr}(Wu_o)(W\delta u)') \\ J_2 &= E(\mathrm{tr}(V\delta y)(V\delta y)' + \mathrm{tr}(W\delta u)(W\delta u)') \ . \end{aligned} \tag{2.6}$$

The goal is now to select $\mathcal{R}$ so that J_1 vanishes. Then, the regulator (2.1) is optimal; no perturbation $n(t)$ could improve the performance, since J_o does not depend on $n(t)$ and $J_2 \geq 0$.

Note that the condition $J_1 = 0$ can be expressed as

$$E((V\delta y)'(W\delta u)')\begin{pmatrix} Vy_o \\ Wu_o \end{pmatrix} = 0 \ .$$

The vector $((Vy_o)'(Wu_o)')'$ contains signals appearing in the criterion when the regulator (2.1) is used. It is required to be orthogonal to the vector of perturbations $((V\delta y)'(W\delta u)')$, caused by admissible variations of the control law.

Assume $y_o(t)$, $\delta y(t)$, $u_o(t)$ and $\delta u(t)$ to be stationary. (This has to be verified, in each particular problem.) Let ℓ be the dimension of the noise vector disturbing the system. By using Parsevals formula, $J_1 = 0$ can be expressed as

$$J_1 = \frac{1}{2\pi i}\oint_{|z|=1} \mathrm{tr}\mathcal{M}(z,z^{-1})\frac{dz}{z} = 0 \tag{2.7}$$

where $\mathcal{M}(z,z^{-1})$ is a rational $\ell|\ell$ matrix. The relation (2.7) is fulfilled if *each element of* $\mathcal{M}(z,z^{-1})z^{-1}$ *is made analytic in* $|z| \leq 1$. Then, the scalar integrand $\mathrm{tr}\mathcal{M}(z,z^{-1})z^{-1}$ is also analytic in $|z| \leq 1$, so the integral vanishes. These ℓ^2 elementwise conditions will be sufficient to determine $\mathcal{R}$. By using matrix fraction descriptions, (MFD's), they can be satisfied collectively, as will be evident in Section III and IV. This results in the same linear polynomial matrix equations as those obtained by the "completing the squares"–part of Kučera's approach.

The outlined procedure is a *constructive* derivation technique. Its initial steps are related to a proof by contradiction, first presented in [10]. In that approach, the optimality of a filter or regulator, obtained by other means, is verified. It is demonstrated that (2.7) is fulfilled, because the integrand is analytic in $|z| \leq 1$. Such non–constructive proofs have been utilized in, for example, [7], [11]–[14].

3 OPEN–LOOP CONTROL: DISTURBANCE MEASUREMENT FEEDFORWARD

Assume a stable system to be described by a model in right MFD form

$$y(t) = BA^{-1}u(t) + DF^{-1}w(t) \quad . \tag{3.1}$$

Here, $w(t) \in R^\ell$ is a vector of stationary measurable disturbances, with spectral density $\phi_w(e^{i\omega})$, described by

$$w(t) = GH^{-1}v(t) \tag{3.2}$$

where $v(t) \in R^\ell$ is stationary white noise, with zero mean and covariance matrix $\psi \geq 0$. The polynomial matrices $B(q^{-1})$, $A(q^{-1})$, $D(q^{-1})$, $F(q^{-1})$, $G(q^{-1})$ and $H(q^{-1})$ have dimensions $p|m$, $m|m$, $p|\ell$, $\ell|\ell$, $\ell|\ell$ and $\ell|\ell$, respectively. Delays are included in the corresponding polynomials of $B(q^{-1})$ and $D(q^{-1})$. The pairs (B,A), (D,F) and (G,H) need not necessarily be right coprime. Assume the following:

A1. The polynomial matrices A, F, G and H are all stable and have nonsingular leading coefficient matrices.

A2. There exists a stable $m|m$ right polynomial spectral factor $\beta(q^{-1})$, defined by

$$\beta_*\beta = B_*V_*VB + A_*W_*WA \tag{3.3}$$

with $\beta(0)$ nonsingular.[1]

In order to minimize (2.2), the perturbed feedforward regulator

$$u(t) = -\mathcal{R}w(t) + n(t) \tag{3.4}$$

is introduced. We assume $w(t)$, but not $y(t)$, to be measurable. All admissible variations can then be expressed as $n(t) = \mathcal{T}w(t)$. The rational matrix $\mathcal{T}(q^{-1})$ must be causal and stable, but is otherwise arbitrary. When the system is controlled by (3.4), outputs and inputs are given by (2.4), where

$$\begin{aligned} y_o(t) &= (DF^{-1} - BA^{-1}\mathcal{R})w(t) \quad ; \quad \delta y(t) = BA^{-1}\mathcal{T}w(t) \\ u_o(t) &= -\mathcal{R}w(t) \quad ; \quad \delta u(t) = \mathcal{T}w(t) \ . \end{aligned} \tag{3.5}$$

The signals are stationary for stable $\mathcal{R}$, since $\mathcal{T}$, F^{-1} and A^{-1} are assumed stable. The use of (3.5) in (2.6) gives

$$\begin{aligned} J_1 &= \frac{1}{2\pi i}\oint_{|z|=1} \mathrm{tr}[V(DF^{-1} - BA^{-1}\mathcal{R})\phi_w\mathcal{T}_*A_*^{-1}B_*V_*]\frac{dz}{z} \\ &\quad -\frac{1}{2\pi i}\oint_{|z|=1} \mathrm{tr}[W\mathcal{R}\phi_w\mathcal{T}_*W_*]\frac{dz}{z} \ . \end{aligned} \tag{3.6}$$

By using $\mathrm{tr}\mathcal{A}\mathcal{B}_* = \mathrm{tr}\mathcal{B}_*\mathcal{A}$, the matrices in both terms get equal dimension $\ell|\ell$. The use of (3.3) and of $\phi_w = GH^{-1}\psi H_*^{-1}G_*$ from (3.2) then gives

$$J_1 = \frac{1}{2\pi i}\oint \mathrm{tr}[\mathcal{T}_*A_*^{-1}(B_*V_*VDF^{-1}G - \beta_*\beta A^{-1}\mathcal{R}G)H^{-1}\psi H_*^{-1}G_*]\frac{dz}{z} \ . \tag{3.7}$$

When (3.7) is set to zero, it corresponds to (2.7). Note that F^{-1}, A^{-1} and H^{-1} have elements with poles only in $|z| < 1$, since they are stable, while V, D, G, β and $1/z$ contribute poles at the origin. They must *all* be eliminated. For that purpose, introduce a right coprime factorization

$$F^{-1}G = G_2F_2^{-1} \tag{3.8}$$

[1]Two conditions are, together, sufficient for A2 to be fullfilled.

1) The matrix $[B_*V_* \ \ A_*W_*]$ has full (normal) row rank m. This is a condition for the existence of a spectral factor [3]. It is fulfilled, for example, if all m inputs are penalized.

2) The greatest common left divisor of B_*V_* and A_*W_* has nonzero determinant on $|z| = 1$. This assures $\det\beta(z^{-1}) \neq 0$ on $|z| = 1$. The factor β is unique, up to a left orthogonal factor.

with G_2 and F_2 both of dimension $\ell|\ell$. Since F^{-1} is stable and causal, so is F_2^{-1}. The position of $\mathcal{R}$ in (3.7) enables direct cancellation of βA^{-1} if $A\beta^{-1}$ is a left factor of $\mathcal{R}$. With $F_2^{-1}G^{-1}$ as a right factor of $\mathcal{R}$, G is also eliminated, while F_2^{-1} can be factored out to the right, to be cancelled later. The regulator becomes

$$\mathcal{R} = A\beta^{-1}QF_2^{-1}G^{-1} \tag{3.9}$$

which is stable and causal, since β^{-1}, F_2^{-1} and G^{-1} are stable and causal. The $m|\ell$ polynomial matrix $Q(z^{-1})$ is not yet specified. Thus,

$$J_1 = \frac{1}{2\pi i}\oint \mathrm{tr}[\mathcal{T}_* A_*^{-1}(B_*V_*VDG_2 - \beta_*Q)F_2^{-1}H^{-1}\psi H_*^{-1}G_*]\frac{dz}{z} \quad . \tag{3.10}$$

The elements of $\mathcal{T}_*(z)A_*^{-1}(z)$ and $H_*^{-1}(z)G_*(z)$ have poles strictly outside $|z|=1$, since $\mathcal{T}$, A and H are stable. Thus, the integrand is made analytic inside $|z|=1$ if there exists a polynomial matrix $L_*(z)$, of dimension $m|\ell$, such that

$$(B_*V_*VDG_2 - \beta_*Q)F_2^{-1}H^{-1}\frac{1}{z} = L_* \tag{3.11}$$

or

$$\beta_*Q + L_*zHF_2 = B_*V_*VDG_2 \quad . \tag{3.12}$$

This is a bilateral Diophantine equation in $Q(z^{-1})$ and $L_*(z)$. Thus, the regulator can be obtained by solving (3.3) for β, computing G_2 and F_2 from (3.8), solving (3.12) for Q and L_*, and using the control law $u(t) = -A\beta^{-1}QF_2^{-1}G^{-1}w(t)$. The reasoning from (3.5) up to equation (3.12) constitutes a simple derivation of the optimal control law.

Remarks. In SISO problems, with $V=1$, $W=\rho\tilde{\Delta}(q^{-1})$, $G=G_2$ and $A=F=F_2$, (3.9) is given by $\mathcal{R}=Q/\beta G$, while (3.12) reduces to equation (3.12) in [11].

The single Diophantine equation (3.12) determines the regulator uniquely [2]. Since $\det\beta_*(z)$ has zeros strictly outside $|z|=1$, while $\det H(z^{-1})F_2(z^{-1})$ has zeros only inside $|z|=1$, the invariant polynomials of β_* are coprime with all those of HF_2. Thus, a solution $(Q^o(z^{-1}), L_*^o(z))$ to (3.12) always exists. See Lemma 1 of [19]. All solutions can be expressed as $(Q, L_*) = (Q^o - XzHF_2, L_*^o + \beta_*X)$, where the polynomial matrix X is arbitrary [3]. However, causality requires $Q(z^{-1})$ to have only nonpositive powers of z as arguments, while $L_*(z)$ must have no negative powers of z as arguments. Otherwise, it would contribute zeros at the origin in (3.10). Thus, $X=0$ is the only choice, so the solution to (3.12) is unique. ∎

[2]This holds in general for *open–loop* control and estimation problems, if the involved systems are stable or marginally stable. If $\det H(z^{-1})$ had zeros in $|z|>1$, two coupled Diophantine equations would sometimes be required to determine $\mathcal{R}$, and formally assure a finite criterion value J_0. However, such control laws, designed to cancel exponentially increasing disturbances, are of no practical interest.

4 STOCHASTIC FEEDBACK CONTROL

Feedback control for a fairly general model structure is now considered. A multivariable counterpart to the model structure for system identification utilized in [16],[17], is

$$Ay(t) = F^{-1}Bu(t) + D^{-1}Ce(t) \tag{4.1}$$

where $y(t)$, $e(t) \in R^p$ and $u(t) \in R^m$. The polynomial matrices A, F, D and C have dimension $p|p$, while B has dimension $p|m$. The noise $e(t)$ is white, stationary and zero mean. Its covariance matrix $\psi > 0$ has dimension $p|p$. A number of well known simpler model structures, inherent in (4.1), are obtained by setting one or several polynomial matrices to the unit matrix. In the identification context, we assume the considered model structures to be identifiable. A causal feedback regulator $u(t) = -\mathcal{R}y(t)$ of dimension $m|p$ is sought, so that the closed loop system is stable and the criterion (2.2) is minimized.

While a left MFD–form is well suited for identification purposes, a right MFD–form is more appropriate for controller design. Assume that $C\psi$ is stable and has full (normal) rank p. Then, the noise term in (4.1) can be viewed as an extended innovations model,[3] $A^{-1}D^{-1}\alpha\varepsilon(t)$, with α stable and defined by $\alpha\alpha_* = C\psi C_*$.[4] The innovations $\varepsilon(t)$ are normalized so that $E\varepsilon(t)\varepsilon(t)' = I_p$. The model (4.1) can then be converted to

$$y(t) = B_2A_2^{-1}u(t) + C_2A_3^{-1}\varepsilon(t) \tag{4.2}$$

where $B_2A_2^{-1}$ and $C_2A_3^{-1}$ are minimal right MFD's of $A^{-1}F^{-1}B$ and $A^{-1}D^{-1}\alpha$, respectively. Since α is stable, so is C_2. We make the following assumptions.

B1. The leading coefficient matrices $A(0)$, $F(0)$, $D(0)$, $C(0)$, and thus also $A_2(0)$, $C_2(0)$ and $A_3(0)$, are nonsingular. Furthermore, $B(0) = 0$, so $B_2(0) = 0$.

B2. The rational matrices FD^{-1} and DF^{-1} are stable. Thus, there may exist unstable right factors D_u of D, $D = D_sD_u$, but they must also be right factors of F, $F = F_sD_u$, while D_s and F_s are stable.

B3. The polynomial matrices FA and B have no unstable common left factors.

B4. There exists a stable $m|m$ right spectral factor, $\beta(q^{-1})$, defined by

$$\beta_*\beta = B_{2*}V_*VB_2 + A_{2*}W_*WA_2 \tag{4.3}$$

with $\beta(0)$ nonsingular. (Cf Footnote 1.)

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[3] An innovations model and its inverse are normally defined stable. Here, it is extended to contain unstable factors.

[4] White measurement noise $w(t)$, with covariance matrix $\phi > 0$, is often present at the output. In other words, $A^{-1}D^{-1}Ce(t) = A^{-1}D^{-1}\alpha\varepsilon(t) = A^{-1}D^{-1}\tilde{C}v(t) + w(t)$. In such a case, full rank p is automatically fulfilled and α is a spectral factor defined by $\alpha\alpha_* = \tilde{C}\tilde{\psi}\tilde{C} + AD\phi D_*A_*$ so that $\alpha(0)$ is nonsingular. If the greatest common left divisor of $\tilde{C}\tilde{\psi}$ and $AD\phi$ has nonzero determinant on $|z| = 1$, α is stable.

Remarks. Unstable modes present only in the noise description could never be stabilized. Unstable modes appearing only in the deterministic subsystems make straightforward LQG–design impossible. Both these cases are excluded by B2 above. It implies that D_u could as well be included in A. Assumptions B2 and B3 together imply stabilizability and detectability of (4.1). Assumption B1 means that the models (4.1) and (4.2) are causal, with the deterministic subsystems being strictly causal. The disturbance models are causally invertible. ■

The innovation sequence $\varepsilon(t)$ in (4.2) is stationary. At time t, it represents the most recent information. Whatever could possibly be achieved by a variation of the regulator, could as well be achieved by adding to the regulator (2.1) a *potential feedforward from the innovations*, $n(t) = \mathcal{T}\varepsilon(t)$. Such a control variation preserves stability, since the feedback loop is unaffected. Hence, introduce the perturbed regulator

$$u(t) = -\mathcal{R}y(t) + n(t) \quad ; \quad n(t) = \mathcal{T}\varepsilon(t) \tag{4.4}$$

where $\mathcal{T}(q^{-1})$ is an arbitrary, but stable and causal $m|p$ transfer matrix. In order to obtain simple expressions for the closed–loop system, the relations

$$\begin{aligned} (I_p + B_2A_2^{-1}\mathcal{R})^{-1}B_2 &= B_2(A_2^{-1}\mathcal{R}B_2 + I_m)^{-1} \\ (I_m + \mathcal{R}B_2A_2^{-1})^{-1}\mathcal{R} &= \mathcal{R}(I_p + B_2A_2^{-1}\mathcal{R})^{-1} \end{aligned} \tag{4.5}$$

which are straighforward to verify, are introduced, as well as the $p|p$ and $m|m$ transfer matrices

$$\mathcal{Y}_o = (I_p + B_2A_2^{-1}\mathcal{R})^{-1}C_2A_3^{-1} \quad , \quad \mathcal{Y}_1 = (\mathcal{R}B_2 + A_2)^{-1} \ . \tag{4.6}$$

The signals $y(t)$ and $u(t)$ of the closed loop system ((4.2) controlled by (4.4)) are then given by (2.4), where

$$\begin{aligned} y_o(t) &= \mathcal{Y}_o\varepsilon(t) & \delta y(t) &= B_2\mathcal{Y}_1\mathcal{T}\varepsilon(t) \\ u_o(t) &= -\mathcal{R}\mathcal{Y}_o\varepsilon(t) & \delta u(t) &= A_2\mathcal{Y}_1\mathcal{T}\varepsilon(t) \ . \end{aligned} \tag{4.7}$$

Following the lines of Section II, an optimal feedback regulator is sought, so that the closed loop system is stable and causal. If such a regulator exists, all transfer functions in (4.7) are stable. (The stability will be verified later, after obtaining the optimal regulator.) The cross term J_1 in (2.5) can then be evaluated in the frequency domain. Invoking (4.7), and using the trace rotation $\mathrm{tr}\mathcal{A}\mathcal{B}_* = \mathrm{tr}\mathcal{B}_*\mathcal{A}$, we obtain

$$\begin{aligned} J_1 &= E\{\mathrm{tr}(V\mathcal{Y}_o\varepsilon(t))(VB_2\mathcal{Y}_1\mathcal{T}\varepsilon(t))' - \mathrm{tr}(W\mathcal{R}\mathcal{Y}_o\varepsilon(t))(WA_2\mathcal{Y}_1\mathcal{T}\varepsilon(t))'\} \\ &= \frac{1}{2\pi i}\oint\{\mathrm{tr}V\mathcal{Y}_oI_p\mathcal{T}_*\mathcal{Y}_{1*}B_{2*}V_* - \mathrm{tr}W\mathcal{R}\mathcal{Y}_oI_p\mathcal{T}_*\mathcal{Y}_{1*}A_{2*}W_*\}\frac{dz}{z} \\ &= \frac{1}{2\pi i}\oint \mathrm{tr}\mathcal{T}_*\mathcal{Y}_{1*}(B_{2*}V_*V - A_{2*}W_*W\mathcal{R})\mathcal{Y}_o\frac{dz}{z} \ . \end{aligned} \tag{4.8}$$

Since $\mathcal{T}$, $\mathcal{Y}_1$ and $\mathcal{Y}_o$ are all assumed stable, $\mathcal{Y}_o$ contributes poles in $|z| < 1$, while no element of $\mathcal{T}_*$ and $\mathcal{Y}_{1*}$ has any pole in $|z| \leq 1$. Hence, to fulfill (2.7), we require

$$(B_{2*}V_*V - A_{2*}W_*W\mathcal{R})\mathcal{Y}_o\frac{1}{z} = L_* \tag{4.9}$$

where $L_*(z)$ is an $m|p$ polynomial matrix. By inspecting (4.9) and consulting (4.3) we find that $\deg L_* \leq \deg \beta_* - 1$. Up to this point, the derivation procedure has followed the same lines as in Section III, where the counterpart to (4.9) determined the polynomial matrix equation directly, cf (3.11). Here, we need some additional steps since, being present also inside $\mathcal{Y}_o$, $\mathcal{R}$ occurs nonlinearly in (4.9). We strive to attain a polynomial matrix relation. Hence, postmultiply (4.9) by $z\mathcal{Y}_o^{-1} = zA_3C_2^{-1}(I_p + B_2A_2^{-1}\mathcal{R})$, express $\mathcal{R}$ as a right MFD, $\mathcal{R} = YX^{-1}$ and rearrange the terms, to obtain

$$(B_{2*}V_*V - zL_*A_3C_2^{-1})X = (A_{2*}W_*W + zL_*A_3C_2^{-1}B_2A_2^{-1})Y \tag{4.10}$$

Observe that $A_3C_2^{-1}B_2A_2^{-1} = \alpha^{-1}DF^{-1}B$ is stable by B2. Thus, we can introduce a minimal right MFD

$$A_3C_2^{-1}B_2A_2^{-1} = B_3C_1^{-1} \tag{4.11}$$

with C_1 being stable. A polynomial matrix relation is obtained from (4.10), if

$$X = C_2X_2 \qquad\qquad Y = C_1Y_2 \quad .$$

Then, (4.10) becomes

$$(B_{2*}V_*VC_2 - zL_*A_3)X_2 = (A_{2*}W_*WC_1 + zL_*B_3)Y_2 \quad .$$

Denote $B_{2*}V_*VC_2 - zL_*A_3 \triangleq Q_1(z, z^{-1})$ and $A_{2*}W_*WC_1 + zL_*B_3 \triangleq Q_2(z, z^{-1})$. Introduce the minimal left coprime MFD $X_1^{-1}Y_1$ as

$$X_1^{-1}Y_1 = Y_2X_2^{-1} = Q_2^{-1}Q_1 \quad .$$

Coprimeness implies that there must exist a polynomial matrix γ, such that $Q_2 = \gamma X_1$ and $Q_1 = \gamma Y_1$, or equivalently,

$$A_{2*}W_*WC_1 + zL_*B_3 \;=\; \gamma X_1 \tag{4.12a}$$

$$B_{2*}V_*VC_2 - zL_*A_3 \;=\; \gamma Y_1 \quad . \tag{4.12b}$$

It remains to determine γ. Postmultiply (4.12a) by $C_1^{-1}A_2$, (4.12b) by $C_2^{-1}B_2$ and add them. Invoking (4.3) and (4.11), we thus obtain

$$\beta_*\beta = \gamma(X_1C_1^{-1}A_2 + Y_1C_2^{-1}B_2) \tag{4.13}$$

or

$$I_m = \beta^{-1}\beta_*^{-1}\gamma(X_1C_1^{-1}A_2 + Y_1C_2^{-1}B_2) \quad . \tag{4.14}$$

Stability of the closed loop is assured if $\gamma = \beta_*$ [5], since with $\mathcal{M}_1 \triangleq \beta^{-1}X_1C_1^{-1}$ and $\mathcal{N}_1 \triangleq \beta^{-1}Y_1C_2^{-1}$, (4.14) is the well known Bezout identity, [2],[3]. Obviously, $\mathcal{M}_1$ and $\mathcal{N}_1$ are stable transfer matrices, free of unstable hidden modes, since C_1, C_2 and β are stable. The Bezout identity guarantees the stability of $(FA + B\mathcal{R})^{-1}$, $\mathcal{R}(FA + B\mathcal{R})^{-1}$, $B_2\mathcal{Y}_1$ and $A_2\mathcal{Y}_1$. See e.g. [3], chapter 5. Since

$$\mathcal{Y}_0 = (I_p + A^{-1}F^{-1}B\mathcal{R})^{-1}A^{-1}D^{-1}\alpha = (FA + B\mathcal{R})^{-1}FD^{-1}\alpha$$

and FD^{-1} is assumed stable by B2, all transfer functions in (4.7) are stable.

Summing up, the regulator is given by

$$\mathcal{R} = YX^{-1} = C_1Y_2X_2^{-1}C_2^{-1} = C_1X_1^{-1}Y_1C_2^{-1} \tag{4.15}$$

where X_1 and Y_1, together with L_*, are the solution of two coupled bilateral polynomial matrix equations (obtained by inserting $\gamma = \beta_*$ into (4.12))

$$\begin{aligned} \beta_*X_1 - zL_*B_3 &= A_{2*}W_*WC_1 \\ \beta_*Y_1 + zL_*A_3 &= B_{2*}V_*VC_2 \ . \end{aligned} \tag{4.16}$$

Remarks. The result above corresponds to Theorem 6.12 in [3], for the case $F = D = I_p$. (Multiply (4.16) by $z^{-n\beta}I_m$, let V and W be constant matrices and substitute Z for $z^{-n\beta}(zL_*)$, E for β, D_1 for C_1 and D_2 for C_2.) In SISO problems, $C_1 = C_2 = C$, $A_2 = AF$ and $A_3 = AD$.

The reasoning above leads to the answer along a short and direct route. One reason is that the variational approach, utilizing a potential feedforward from the innovations, quickly leads to a condition (4.9) for optimality. Another reason is that the two coupled design equations (4.12a),(4.12b) are obtained simultaneously. In the "completing the squares" approach, these equations are derived sequentially. The suggested method becomes very similar in other feedback control problems, such as the optimization of state feedbacks using the polynomial approach [5].

Solvability of (4.16) can be proved as in [3]. Since $Z = z^{-n\beta+1}L_*$ and $\deg L_* = \deg\beta - 1$, it is clear that $\deg Z < \deg\overline{\beta}$ is fulfilled. Note that the free z in (4.16), originating from the $1/z$ in (2.7), is the reason for this strict inequality.

Controllers with several degrees of freedom can be optimized with the proposed approach. Several different variations must then be introduced, one for each degree of freedom. The cross–term (2.5) will then consist of several terms, which should be made zero separately. See [20] for an example.

[5]This will in fact be the only choice for (4.12a) and (4.12b) to be solvable, with respect to $X_1(z^{-1}), Y_1(z^{-1})$ and $L_*(z)$.

5 CONCLUDING REMARKS

A simple derivation methodology for use within the polynomial approach to linear quadratic optimization, has been presented. Its ability to derive solutions, in just a few well defined steps, has been illustrated by means of a feedforward and a feedback problem. Two key ideas in the reasoning can be distinguished.

- By means of Parseval's formula, orthogonality is constructively evaluated in the frequency domain, to obtain the design equations. An important insight here is that the calculations become straightforward in the multivariable case if the whole matrix $\mathcal{M}(z, z^{-1})$ in (2.7) is manipulated, not only its trace. Otherwise, the regulator could not be specified uniquely.
- Feedback regulators are optimized by regarding the variational term $n(t)$ as a *potential feedforward* from the innovations. If that cannot improve the control performance, nothing else can. Internal stability then becomes easy to ascertain.

The suggested approach can be summarized as a step–by–step scheme as follows.

1. Convert the system description to right MFD–form and introduce a criterion–related right polynomial matrix spectral factorization. Introduce a controller perturbation. Obtain expression (2.4) for the perturbed controlled system.
2. Evaluate the crossterm J_1 of the criterion (2.5). Fulfill (2.7), $J_1 = 0$, by requiring $\mathcal{M}(z, z^{-1})z^{-1} = \mathcal{L}_*(z)$, where elements of $\mathcal{L}_*$ have poles only in $|z| > 1$. Ascertain this by means of polynomial (not rational) matrix equations. In *open loop* problems of practical interest, the result is a single Diophantine equation.
3. In *feedback problems*, parametrize the regulator in right MFD–form, $\mathcal{R} = YX^{-1}$ (with X_2 and Y_2 right factors of X and Y), so that $\mathcal{M}(z, z^{-1})z^{-1} = \mathcal{L}_*$ can be converted to a polynomial matrix relation of the form

$$Q_1(z, z^{-1})X_2(z^{-1}) = Q_2(z, z^{-1})Y_2(z^{-1}) \quad .$$

   By representing $Y_2X_2^{-1}$ as a minimal left MFD, two coupled Diophantine equations, determining the regulator, are obtained.

While Step 2 concludes an open loop controller derivation, the additional Step 3 is needed in feedback problems, mainly since $\mathcal{R}$ appears nonlinearly in the relation $\mathcal{M}(z, z^{-1})z^{-1} = \mathcal{L}_*$. Use of the Youla–Kučera parametrization is not required.

The methodology has been demonstrated on (slight generalizations of) well-known problems with known solutions. The purpose of the proposed method is, of course, to facilitate the derivation of novel results. It has been used in [20] to derive e.g. a MIMO combined feedback and feedforward regulator, and in [21] to derive design equations for robust estimators and feedforward controllers.

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