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Feedforward control is dual to deconvolution

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A duality is demonstrated between optimal feedforward control and optimal deconvolution, or input estimation. These two problems are normally discussed separately in the literature, but have close similarities. Duality between them can be demonstrated if and only if one uses general problem formulations, with frequency-shaped weighting in the criteria. From one of the problems, the dual problem can then be obtained immediately from the block diagram, by reversing the directions of arrows, interchanging summation points and node points and transposing all transfer function matrices. This result applies for continuous and discrete time problems, as well as for minimization of $J = \|G\|$, for any transfer function norms for which $\|G^T\| = \|G\|$. A derivation of a polynomial solution to the frequency-weighted discrete-time MIMO LQG feedforward control problem illustrates the use of the duality.

1. Introduction

Duality relations have a long history as fruitful tools in control and estimation theory. All control engineers are well aware of the dualities between LQ state feedback and Kalman state estimation (see Kalman 1960 or, e.g., Kwakernaak and Sivan, 1972). Similar duality results are of use in the study of H_∞-control and state estimation (see Doyle *et al.* 1989 and Shaked, 1990). For a linear time-invariant system in state-space form, the dual system is obtained by reversing the role of inputs and outputs and by transposing all matrices.

For a problem in block diagram form, there is, however, no need to compute a state space realization to obtain the dual problem. In § 2 and 3, we present an elegant way of obtaining the dual of a linear time-invariant control problem directly, from its block diagram. Although we have not found Theorem 1 in § 2 stated explicitly in the literature, we doubt that it is novel. However, we include it because it is important for the following discussion. Also, the elegant algorithm in § 3 is believed to be a part of the folklore of control theorists, but is hard to find in the literature. The result is, however, very useful for example when doing polynomial calculations and has certainly not greatly penetrated the literature.

This method is used in § 4–6 to clarify the question of what kind of estimation problems are dual to feedforward control problems. This question has been discussed for example in Sternad and Ahlén (1988), where close correspondences were pointed out between disturbance measurement feedforward control and deconvolution, also called input estimation. In fact, by using loop transformations on scalar problems, it was shown how one problem could be transformed into the other. No dual relationship could, however, be obtained.

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As will be clarified below, it is possible to demonstrate a dual relationship, if the formulations of both problems are made more general than the ones discussed in Sternad and Ahlén (1988). The general formulations include *dynamic cost weighting* in the criteria.

Our interest has been mainly in LQG (or H_2)-solutions based on polynomial equations, a method pioneered by Kučera (1979). However, the duality holds for any criterion $J = \min \|G\|$ based on a norm of a rational matrix G, for which $\|G^{\mathrm{T}}\| = \|G\|$. It does for example hold also for an H_∞ norm but not in the MIMO case for the L_1 norm (Dahleh and Pearson 1987), defined by

$$\|G\|_{L_1} = \max_i \sum_{j=1}^{n} \|g_{ij}\|_1$$

where $g_{ij}(t)$ is the impulse-response of element (i, j) of $G(s)$.

By clarifying the duality relation between the two types of problems, we achieve two goals. Firstly, the many correspondences between them are explained, and the understanding of both problems is enhanced, see § 6 and 8. Intuition from the feedforward problem can be used in the formulation and solution of input estimation problems and vice versa. Secondly, the construction of algorithms for computer-aided design is simplified. Only one algorithm, which solves both kinds of problems, needs to be implemented. We illustrate this in § 7 by deriving a polynomial solution to the discrete time LQG feedforward control problem from the corresponding input estimator design equations.

2. Duality

We begin our discussion by establishing a duality relation between the two problems described by the block diagrams in Fig. 1. The left-hand diagram in Figure 1 represents the 'standard problem'. It was introduced around 1980 as a standard way of representing a large collection of control and signal estimation problems (see e.g. Pernebo 1981 or Doyle *et al.*, 1989). Polynomial optimization of LQG-controllers for the standard problem is described in Grimble (1991) and Hunt *et al.* (1991).

In Fig. 1, y represents the measured variables, z are signals to be controlled, w are exogenous signals and u are the control inputs. Many control and filtering problems are formulated as the design of K_1, to minimize the influence of w on

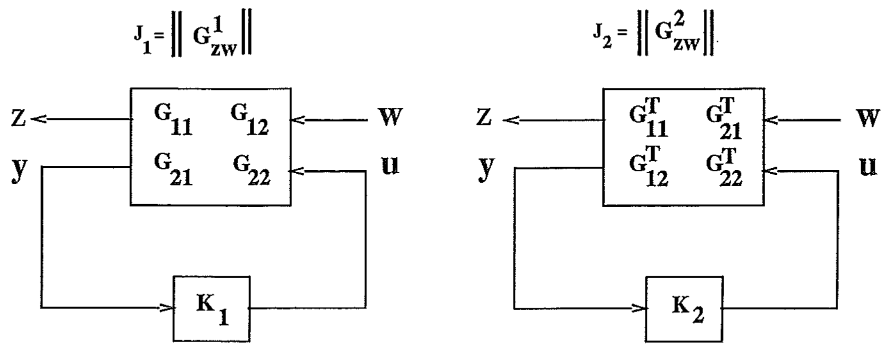

Figure 1. Dual problems: the left-hand figure represents the standard problem; the dual problem is given to the right.

z. All G_{ij} are here linear time-invariant transfer functions, in continuous or discrete time. (Time-arguments of signals, and arguments of transfer functions, are suppressed in the following.) Duality between the two block diagrams in Fig. 1 can now be stated as follows.

Theorem 1. Problem duality: *For all norms satisfying* $\|G^{\mathrm{T}}\| = \|G\|$, *the two problems*

$$J_1 = \min_{K_1} \|G^1_{zw}\|, \quad J_2 = \min_{K_2} \|G^2_{zw}\|$$

in Fig. 1 are dual, in the sense that the two optima are equal $J_1 = J_2$, *and the optimal controllers are related by* $K_1^{\mathrm{T}} = K_2$. *A necessary condition for the problems to be dual is that the minimal values of the norms are invariant under transposition.*

Proof: The closed loop from w to z in the first problem is given by

$$G^1_{zw} = G_{11} + G_{12}(I - K_1 G_{22})^{-1} K_1 G_{21}$$

Transposing gives

$$\begin{aligned}(G^1_{zw})^{\mathrm{T}} &= G_{11}^{\mathrm{T}} + G_{21}^{\mathrm{T}} K_1^{\mathrm{T}} (I - G_{22}^{\mathrm{T}} K_1^{\mathrm{T}})^{-1} G_{12}^{\mathrm{T}} \\ &= G_{11}^{\mathrm{T}} + G_{21}^{\mathrm{T}} (I - K_1^{\mathrm{T}} G_{22}^{\mathrm{T}})^{-1} K_1^{\mathrm{T}} G_{12}^{\mathrm{T}}\end{aligned}$$

which is exactly the closed loop from w to z in the second case if $K_2 = K_1^{\mathrm{T}}$. The sufficiency of the assumption $\|G^{\mathrm{T}}\| = \|G\|$ follows from

$$\|G^1_{zw}\| = \|(G^1_{zw})^{\mathrm{T}}\| = \|G^2_{zw}\|$$

If $\|G^1_{zw}\| \neq \|(G^1_{zw})^{\mathrm{T}}\|$ at the minimum, then J_1 and J_2 will differ, Thus, it is necessary that $\|G^1_{zw}\| = \|(G^1_{zw})^{\mathrm{T}}\|$ at the minimum. □

Remark 1: The dual system can be obtained by transposing the following matrix, where the state-space representation of the transfer functions G_{ij} is $[A, B_j, C_i, D_{ij}]$, $i, j = 1, 2$:

$$\begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}$$ □

Remark 2: Depending on the specific type of problem set-up and norm, restrictions may have to be imposed on the properness and stability of some or all of the blocks G_{ij}. □

Remark 3: Fundamentally, duality is a relation between two *systems*: the role of their inputs and outputs are interchanged, and the time is reversed. For a time varying, continuous time system, the transformation can be seen as obtaining the adjoint system, followed by a time reversal. With finite final time t_f, the transformation of the state-space description is

$$\begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \xrightarrow{\text{Adjoint system}} \begin{bmatrix} -A^{\mathrm{T}}(t) & -C^{\mathrm{T}}(t) \\ B^{\mathrm{T}}(t) & D^{\mathrm{T}}(t) \end{bmatrix} \xrightarrow{\text{Time reversal}} \begin{bmatrix} A^{\mathrm{T}}(t_f - t) & C^{\mathrm{T}}(t_f - t) \\ B^{\mathrm{T}}(t_f - t) & D^{\mathrm{T}}(t_f - t) \end{bmatrix}$$

The last transformation follows because the solution to $\dot{v} = -f(t_f - t, v(t))$ is the time reverse of the solution to $\dot{x} = f(t, x(t))$, see e.g. Kwakernaak and

Sivan (1972), Lemma 4.1. (The dual system is identical to the so-called *modified adjoint system*, see Kailath 1980). With time-invariant systems, the transformation (1) reduces to a transposition.

Duality between systems can be used for obtaining correspondences, or dualities, between optimization problems. The original example is LQ state feedback and Kalman filtering (Kalman 1960). If the solution is time-invariant, we no longer have to think of the dual problem as defined in reversed time. Thoeorem 1 states that duality essentially involves only transposition for a large class of problems with time-invariant solutions.

3. Block diagram version of duality

For time invariant systems given in block diagram form, the dual to an optimization problem, in the sense of Theorem 1 can, if it exists, be obtained directly from the block diagram. The idea is old, but has to the author's knowledge not been published for the general set-up of Theorem 1. The correctness of the following algorithm is easily proved and is left as a nice exercise.

Algorithm: The following block diagram transformations give the dual block diagram.

$$\left.\begin{array}{l}\textit{Step } 1 \text{ Exchange } w\text{:}s \text{ and } z\text{:}s.\\ \textit{Step } 2 \text{ Exchange } u\text{:}s \text{ and } y\text{:}s.\\ \textit{Step } 3 \text{ Reverse directions of arrows.}\\ \textit{Step } 4 \text{ Interchange summation points and node points.}\\ \textit{Step } 5 \text{ Transpose the transfer function blocks.}\end{array}\right\} \quad (2)$$

An example is given in Fig. 2, and also in Fig. 4 below. □

4. Feedforward control

The design of feedforward links from measurable disturbances and from command signals is an important complement to a feedback design. We will here, in particular, consider the design of LQG (or H_2)-controllers.

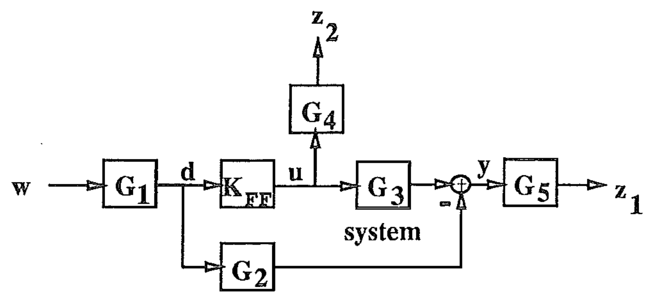

Figure 2. Feedforward control problems.

The feedforward problem to be considered is shown in Fig. 2. The system output is described by

$$y = G_3 u - G_2 d$$

where G_3 represents the system, including a possible fixed feedback controller. Here, d is a measurable signal, which is modelled as filtered white noise.

$$d = G_1 w$$

The problem is to calculate the optimal causal, stable and linear feedforward regulator

$$u = K_{\mathrm{FF}} d$$

which minimizes a mean square of the sum of filtered outputs and filtered control signals:

$$\min \mathrm{E}\,(\operatorname{tr} z_1 z_1^{\mathrm{T}} + \operatorname{tr} z_2 z_2^{\mathrm{T}})$$

$$z_1 = G_5 y$$

$$z_2 = G_4 u$$

All transfer functions are assumed known, stable, proper (in continuous time) or causal (in discrete time). In discrete-time problems, both G_2 and G_3 may include delays. In a disturbance measurement feedforward problems, d represents measurable disturbances. These are eliminated in frequency regions of interest (defined by G_5), if $z_1 = G_5(G_3 K_{\mathrm{FF}} - G_2) G_1 w$ is small. In command feedforward problem, d represents command signals, and $G_1 w$ are stochastic models describing their second-order properties. Servo filters K_{FF} are then to be designed, based on a response model G_2. Good model following is achieved, in frequency regions of interest, if z_1 is small. (In a multivariable setting, d can of course include both measurable disturbances and command signals.)

For a discussion of scalar discrete-time LQG feedforward design, see e.g. Sternad and Söderström (1988) or Hunt (1989). Multivariable problems are discussed in Hunt and Šebek (1989) and Sternard and Ahlén (1992), using the polynomial equations approach. A solution to MIMO discrete time problems is discussed in § 7.

5. Estimation of the input to a dynamic system

Many filtering, prediction and smoothing problems are special cases of the set-up presented in Fig. 3. The signal u is the input to a linear system G_3^{T}. A possibly filtered version of it, $u_{\mathrm{f}} = G_2^{\mathrm{T}} u$, is to be estimated, based on noisy measurements y of the system output. With white noise w_1 and w_2, $G_5^{\mathrm{T}} w_1$ and $G_4^{\mathrm{T}} w_2$ represent stochastic models of signal and noise. The transfer function G_1^{T} is a frequency shaping weighting filter.

When G_3^{T} contains dynamic elements, the problem is an input estimation or deconvolution problem. Otherwise, we have an output or state estimation problem. A dynamic element G_3^{T} may represent an analogue or digital communication channel. The filter K_{E} is then a linear recursive equalizer. Its task is to reconstruct the transmitted signal u. In process control and supervision, G_3^{T} can represent a transducer, with slow dynamics. The task of the filter K_{E} is then to estimate the input u to the transducer.

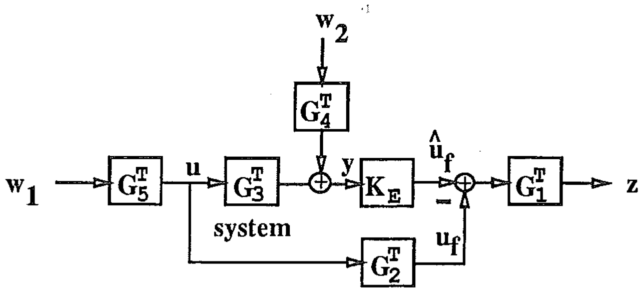

Figure 3. Signal estimation; the filter K_{E} is sought which estimates (a filtered version of) the input u.

The filter G_2^{T} in the lower (fictitious) signal path can be of use in several ways. In discrete time, it may include an advance or delay Iq^{-m} i.e. $u_{\mathrm{f}}(t) = u(t-m)$. Depending on m, $\hat{u}(t-m|t)$ is then a prediction ($m<0$), filtering ($m=0$) or a fixed lag smoothing ($m>0$) estimate. The block G_2^{T} may also contain filters, to emphasize the estimation accuracy in certain frequency regions. Filters in either G_1^{T} or G_2^{T} can be used for affecting the relative accuracy, in different frequency regions, in the estimation of u. For a discussion of advantages and disadvantages of these two methods, see Ahlén and Sternad (1989). Thus, the measured output is described by

$$y = G_3^{\mathrm{T}}u + G_4^{\mathrm{T}}w_2; \quad u = G_5^{\mathrm{T}}w_1$$

All systems are assumed to be known and stable and the white noise signals w_i are stationary, zero mean and mutually uncorrelated. We consider the problem of finding the best causal, stable and linear estimator of a filtered version $G_2^{\mathrm{T}}u$ of the input

$$\hat{u}_{\mathrm{f}} = K_{\mathrm{E}}y$$

which minimizes a frequency weighted version of the mean square estimation error

$$\min \mathrm{E}\,(\mathrm{tr}\, zz^{\mathrm{T}}); \quad z = G_1^{\mathrm{T}}(\hat{u}_{\mathrm{f}} - G_2^{\mathrm{T}}u)$$

Such an estimator is a Wiener or stationary Kalman filter. All blocks, except G_2^{T}, are assumed proper in continuous time and causal in discrete time.

A solution to the discrete-time version of the MIMO H_2 estimation problem introduced above will be discussed in § 7.

6. Duality between feedforward control and input estimation

Using the set-up in Fig. 1, the feedforward problem in Fig. 2 is represented within the standard problem by the rational matrix

$$\begin{bmatrix} \begin{bmatrix} -G_5G_2G_1 \\ 0 \end{bmatrix} & \begin{bmatrix} G_5G_3 \\ G_4 \end{bmatrix} \\ G_1 & 0 \end{bmatrix}$$

with z_1, z_2 as controlled outputs, d as measured output, w as exogenous input and u as control input. Transposing this matrix gives the dual problem

$$\begin{bmatrix} [-G_1^T G_2^T G_5^T \ 0] & G_1^T \\ [G_3^T G_5^T \ G_4^T] & 0 \end{bmatrix}$$

This is exactly the estimation problem of Fig. 3, with z as output to be minimized, y as measured output, w_1, w_2 as exogenous inputs, $\hat{u}_f$ as 'control input', and $K_E = K_{FF}^T$. Alternatively, we may use the block diagram transformations in § 3 on Fig. 2 directly, which gives Fig. 4.

Thus, the model G_1 of the signal d corresponds to the weighting function of the estimation problem. The systems G_3 correspond to each other. The control weighting G_4 corresponds to the measurement noise, while the output weighting G_5 corresponds to the signal model of the estimation problem. As in other dual problems, minimum variance control ($G_4 = 0$) corresponds to estimation with noise-free measurements. In continuous time, both are singular problems. Some of the consequences of these correspondences will be discussed in § 8.

7. An illustration: polynomial solutions to discrete-time input estimation and feedforward control problems

7.1. *Input estimation/deconvolution*

In a discrete-time estimation problem described by Fig. 3, let the noise-corrupted measurement vector $y(t)$, of dimension p, and the input $u(t)$, of dimension s, be given by

$$\left.\begin{aligned} y(t) &= A^{-1}Bu(t) + N^{-1}Mw_2(t) \\ u(t) &= D^{-1}Cw_1(t) \end{aligned}\right\} \tag{3}$$

Here, (A, B, N, M, D, C) are polynomial matrices in the backward shift operator q^{-1}, of dimension $p|p$, $p|s$, $p|p$, $p|r$, $s|s$ and $s|k$, respectively. The noise signals $\{w_1(t)\}$ and $\{w_2(t)\}$ are assumed white and stationary, with zero means and covariance matrices normalized to unit matrices. An optimal linear estimator

$$\hat{u}_f(t) = K_E(q^{-1})y(t) \tag{4}$$

of a filtered version of the input, of dimension l

$$u_f(t) = T^{-1}Su(t - m) \tag{5}$$

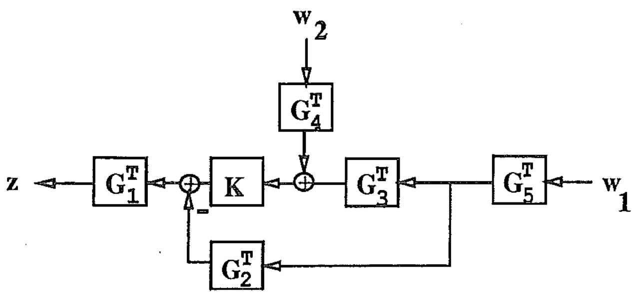

Figure 4. Result of block diagram transformations on the feedforward problem in Fig. 2; the result equals the block diagram of the estimation problem in Fig. 3.

is sought, such that the frequency weighted quadratic criterion

$$J = \operatorname{tr} E\{z(t)z^{\mathrm{T}}(t)\}; \quad z(t) = U^{-1}V(\hat{u}_{\mathrm{f}}(t) - u_{\mathrm{f}}(t)) \tag{6}$$

is minimized. In (5) and (6), T, S, U, V are polynomial matrices of dimensions $l|l$, $l|s$, $l|l$ and $l|l$.

Comparing with Figs 3 or 4, we have

$$G_1^{\mathrm{T}} = U^{-1}V$$
$$G_2^{\mathrm{T}} = T^{-1}Sq^{-m}$$
$$G_3^{\mathrm{T}} = A^{-1}B$$
$$G_4^{\mathrm{T}} = N^{-1}M$$
$$G_5^{\mathrm{T}} = D^{-1}C$$

We make the following two assumptions.

Assumption 1: *The polynomial matrices $A(q^{-1})$, $N(q^{-1})$, $D(q^{-1})$, $T(q^{-1})$, $U(q^{-1})$, and $V(q^{-1})$ all have stable determinants and non-singular leading coefficient matrices. (Thus, they have stable and causal inverses.)*

Assumption 2: *The spectral density matrix $\Phi_y(e^{i\omega})$ of the measurement $y(t)$ is non-singular for all ω.*

Define the following coprime factorizations

$$BD^{-1} = \tilde{D}^{-1}\tilde{B} \tag{7}$$
$$\tilde{D}AN^{-1} = \tilde{N}^{-1}\tilde{A} \tag{8}$$
$$VT^{-1}SD^{-1} = \tilde{T}^{-1}\tilde{S} \tag{9}$$

Stability of $\det T$ and $\det D$ and coprimeness of $\tilde{T}^{-1}\tilde{S}$ implies that $\det \tilde{T}$ will be stable. Causality of T^{-1} and D^{-1} implies that $\tilde{T}^{-1}$ will be causal. Let P_* denote the conjugate transpose $P^{\mathrm{T}}(q)$ of a polynomial matrix $P(q^{-1})$. Define the following left polynomial spectral factorization,

$$\beta\beta_* = \tilde{N}\tilde{B}CC_*\tilde{B}_*\tilde{N}_* + \tilde{A}MM_*\tilde{A}_* \tag{10}$$

Under Assumption 2, (10) will always have a solution $\beta(q^{-1})$, of dimension $p|p$, with stable determinant and non-singular leading coefficient matrix (see, for example, Anderson and Moore 1979, Kučera 1979, 1980, Ježek and Kučera 1985).† The following result now holds.

Theorem 2. The Wiener estimator: *Let the system and input model be described by (3), see Fig. 3. Introduce the coprime factorizations (7)–(9) and the spectral factorization (10). Under Assumptions 1 and 2, a stable and causal H_2-optimal estimator (4), minimizing (6), is then given by*

$$\hat{u}_{\mathrm{f}}(t) = V^{-1}\tilde{T}^{-1}Q_{\mathrm{E}}\beta^{-1}\tilde{N}\tilde{D}Ay(t) \tag{11}$$

†Two conditions on the polynomial matrices appearing in (10) do, together, guarantee that Assumption 2 holds. (1): The matrix $[\tilde{N}\tilde{B}C \ \tilde{A}M]$ should have full (normal) row rank p and (2): the greatest common left divisor of $\tilde{N}\tilde{B}C$ and $\tilde{A}M$ should have a non-zero determinant on $|z| = 1$. While (1) is a condition for existence of spectral factors, (2) provides a spectral factor $\beta(z^{-1})$ such that $\det \beta(z^{-1}) \neq 0$ on $|z| = 1$.

where $Q_E(q^{-1})$, together with $L_*(q)$, both of dimension $l|p$, are given by the unique solution to the bilateral diophantine equation

$$q^{-m}\tilde{S}CC_*\tilde{B}_*\tilde{N}_* = Q_E\beta_* + q\tilde{T}UL_* \tag{12}$$

Proof: In Ahlén and Sternad (1991), this result is derived for the case $U = V = I$. Only small modifications, leading to (9), (11), and (12) above, while (7), (8), (10) remain unchanged, are required to extend that result to filtered criteria $z(t) = U^{-1}V(\hat{u}_f(t) - u_f(t))$. Note that since $\tilde{T}^{-1}$ and β^{-1} are stable and causal, (11) will be stable and causal. □

For a more detailed discussion of Wiener filter design using polynomial equations, see Ahlén and Sternad (1991).

7.2. *Feedforward control*

Let us, in the same way, express a discrete time feedforward structure, described by Fig. 2, by right matrix fraction descriptions:

$$\left.\begin{aligned}
\text{Disturbance/reference dynamics: } G_1 &= G_cH_c^{-1} \\
\text{Disturbance transfer/desired response model: } G_2 &= q^{-m}D_cF_c^{-1} \\
\text{System: } G_3 &= B_cA_c^{-1} \\
\text{Input weighting function: } G_4 &= W_cN_c^{-1} \\
\text{Output weighting function: } G_5 &= V_cU_c^{-1}
\end{aligned}\right\} \tag{13}$$

Assume A_c, N_c, U_c, F_c, H_c and G_c to have stable determinants and non-singular leading coefficient matrices. Introduce the coprime factorizations

$$U_c^{-1}B_c = \tilde{B}_c\tilde{U}_c^{-1} \tag{14}$$

$$N_c^{-1}A_c\tilde{U}_c = \tilde{A}_c\tilde{N}_c^{-1} \tag{15}$$

$$U_c^{-1}D_cF_c^{-1}G_c = \tilde{G}_2\tilde{F}_2^{-1} \tag{16}$$

Stability of $\det U_c$ and $\det F_c$ and coprimeness of $\tilde{G}_2\tilde{F}_2^{-1}$ implies that $\det \tilde{F}_2$ will be stable. Causality of U_c^{-1} and F_c^{-1} implies that $\tilde{F}_2^{-1}$ will be causal. Define the following criterion-related right polynomial spectral factorization

$$\beta_{c*}\beta_c = \tilde{N}_{c*}\tilde{B}_{c*}V_{c*}V_c\tilde{B}_c\tilde{N}_c + \tilde{A}_{c*}W_{c*}W_c\tilde{A}_c \tag{17}$$

Assume that the right-hand side of (17) is non-singular on the unit circle. Then, (17) will have a solution β_c with stable and causal inverse.

Now, the polynomial solution to the LQG feedforward design problem can be stated as follows.

Theorem 3. The LQG feedforward regulator: *Let the system and weighting functions in Fig. 3 be given by the right MFDs (13). Introduce the coprime factorizations (14)–(16) and the spectral factorization (17), non-singular on $|z| = 1$. Then, a stable and causal H_2-optimal feedforward regulator, minimizing*

$$\operatorname{tr} E\{(G_5y(t))(G_5y(t))^T + (G_4u(t))(G_4u(t))^T\}$$

is

$$u(t) = A_cU_c\tilde{N}_c\beta_c^{-1}Q_{FF}\tilde{F}_2^{-1}G_c^{-1}d(t) \tag{18}$$

where $Q_{\mathrm{FF}}(q^{-1})$, together with $L_{1*}(q)$ are given by the unique solution to the bilateral diophantine equation

$$q^{-m}N_{c*}\widetilde{B}_{c*}V_{c*}V_c\widetilde{G}_2 = \beta_{c*}Q_{\mathrm{FF}} + qL_{1*}H_c\widetilde{F}_2 \tag{19}$$

Proof: The solution to this problem will be derived by duality with (3)–(6). Use of the duality relations give (with $P^{-\mathrm{T}}$ denoting the transpose and inverse of P)

$$\begin{aligned}
G_1^{\mathrm{T}} &= H_c^{-\mathrm{T}}G_c^{\mathrm{T}} && \leftrightarrow U^{-1}V \\
G_2^{\mathrm{T}} &= F_c^{-t}D_c^{\mathrm{T}}q^{-m} && \leftrightarrow T^{-1}Sq^{-m} \\
G_3^{\mathrm{T}} &= A_c^{-\mathrm{T}}B_c^{\mathrm{T}} && \leftrightarrow A^{-1}B \\
G_4^{\mathrm{T}} &= N_c^{-\mathrm{T}}W_c^{\mathrm{T}} && \leftrightarrow N^{-1}M \\
G_5^{\mathrm{T}} &= U_c^{-\mathrm{T}}V_c^{\mathrm{T}} && \leftrightarrow D^{-1}C
\end{aligned}$$

By making the substitutions above, and transposing (7)–(12), design equations are obtained for the LQG feedforward regulator. Substitution into (7)–(9) gives

$$\begin{aligned}
B_c^{\mathrm{T}}U_c^{-\mathrm{T}} &= \widetilde{D}^{-1}\widetilde{B} \\
\widetilde{D}A_c^{\mathrm{T}}N_c^{-\mathrm{T}} &= \widetilde{N}^{-1}\widetilde{A} \\
G_c^{\mathrm{T}}F_c^{-\mathrm{T}}D_c^{\mathrm{T}}U_c^{-\mathrm{T}} &= \widetilde{T}^{-1}\widetilde{S}
\end{aligned}$$

By transposing these factorizations, and defining $\widetilde{B} \triangleq \widetilde{B}^{\mathrm{T}}$, $\widetilde{U}_c \triangleq \widetilde{D}^{\mathrm{T}}$, $\widetilde{A}_c \triangleq \widetilde{A}^{\mathrm{T}}$, $\widetilde{N}_c \triangleq \widetilde{N}^{\mathrm{T}}$, $\widetilde{G}_2 \triangleq \widetilde{S}^{\mathrm{T}}$, $\widetilde{F}_2 \triangleq \widetilde{T}^{\mathrm{T}}$, we obtain (14)–(16).

Use of the substitutions and of (14), (15) in the spectral factorization (10) gives

$$\beta\beta_* = \widetilde{N}_c^{\mathrm{T}}\widetilde{B}_c^{\mathrm{T}}V_c^{\mathrm{T}}V_{c*}^{\mathrm{T}}\widetilde{B}_{c*}^{\mathrm{T}}\widetilde{N}_{c*}^{\mathrm{T}} + \widetilde{A}_c^{\mathrm{T}}W_c^{\mathrm{T}}W_{c*}^{\mathrm{T}}\widetilde{A}_{c*}^{\mathrm{T}}$$

By transposing and defining $\beta_c \triangleq \beta^{\mathrm{T}}$, we obtain the criterion-related right spectral factorization (17).

The feedforward filter (18) is obtained by substitution into (11) and transposition:

$$u(t) = (G_c^{-\mathrm{T}}\widetilde{F}_2^{-\mathrm{T}}Q_{\mathrm{E}}\beta^{-1}\widetilde{N}_c^{\mathrm{T}}U_c^{\mathrm{T}}A_c^{\mathrm{T}})^{\mathrm{T}}d(t)$$

By defining $Q_{\mathrm{FF}} \triangleq Q_{\mathrm{E}}^{\mathrm{T}}$, (18) is obtained. The filter will, of course, be stable and causal, since $\widetilde{F}_2^{-1}$ and β^{-1} are stable and causal. Substitution into the diophantine equation (12) gives

$$q^{-m}\widetilde{G}_2^{\mathrm{T}}V_c^{\mathrm{T}}V_{c*}^{\mathrm{T}}\widetilde{B}_{c*}^{\mathrm{T}}\widetilde{N}_{c*}^{\mathrm{T}} = Q_{\mathrm{E}}\beta_{c*}^{\mathrm{T}} + q\widetilde{F}_2^{\mathrm{T}}H_c^{\mathrm{T}}L_* \tag{20}$$

By transposing this equation and using $Q_{\mathrm{FF}} = Q_{\mathrm{E}}^{\mathrm{T}}$ and $L_{1*} \triangleq L_*^{\mathrm{T}}$, we obtain the diophantine equation (19). □

The minimal criterion value will, of course, be equal in the two dual problems.

The feedforward design equations (14)–(19) constitute an extension of earlier known results. In Sternad and Ahlén (1992), only the special case of polynomial penalties in the criterion, $N_c = I$ and $U_c = I$, were considered. (The two coprime factorizations (14), (15) are then superfluous, with $\widetilde{B}_c = B_c$, $\widetilde{A}_c = A_c$.) Hunt and Šebek (1989) consider a different combined feedback and feedforward problem, without dynamic cost weighting.

Remark: The extension to dynamic cost weighting clearly shows how the weights influence the controller. This can help the user in the choice of weights. It should, however, be noted that another possibility is to include the weights in an extended system description, see e.g. Hunt (1989). □

Remark: In both the estimation and the feedforward control problems, one can derive a second diophantine equation. For unstable systems, it would sometimes have to be used in combination with (12)/(19) to determine the filter uniquely. This is never necessary for systems with poles on or inside the stability limit. Since strictly unstable systems are of little relevance in the open-loop design problems considered here, we have not introduced this second equation, which would just complicate the solution. However, the duality relations do, of course, hold for that equation as well. □

8. Concluding discussion

It has been demonstrated that *feedforward control* problems are dual to a special type of estimation problems: *deconvolution* or *input estimation* problems, where the input to a dynamic system G_3^{T} is sought. (Output or state estimation problems, without transducer dynamics G_3^{T}, would correspond to rather trivial feedforward control problems, with no dynamics between control input u and the output y.)

In Sternad and Ahlén (1988), several close correspondences were noted between scalar Wiener-input estimation and LQG feedforward control problems. These correspondences could not be interpreted as duality relations. The reason for this can now be seen in the too restrictive problem formulations used in Sternad and Ahlén (1988): $G_4 = I$ and $G_5 = I$ in the control problem and $G_2^{\mathrm{T}} = I$ and $G_1^{\mathrm{T}} = I$ in the input estimation problem. With duality established between the more general problems discussed in this paper, the correspondences between (MIMO) LQG feedforward controllers and Wiener input estimators can now be placed into their correct perspective. Some design guidelines also follow.

(*a*) When the system G_3 is of low-pass-type, both feedforward controllers and Wiener input estimators tend to be high-pass. In the control problem, an input penalty G_4, penalizing high-frequency components of the input, will reduce the high frequency gain of the controller K_{FF}. The introduction of measurement noise $G_4^{\mathrm{T}} w_2$ with significant high-frequency content has the same effect on the estimator K_{E}. For scalar problems, a resonance peak in G_4 introduces a notch in both K_{FF} and K_{E}. Note the presence of $\tilde{N}$ in (11) and of $\tilde{N}_{\mathrm{c}}$ in (18). They equal N and N_{c}, respectively in scalar problems.

(*b*) Use of a positive smoothing lag m in the estimation problem (with $q^{-m} I$ in G_2^{T}) corresponds to a delay in the disturbance path ($q^{-m} I$ in G_2) of a regulator problem. A larger smoothing lag/delay will improve the filtering/control performance.

(*c*) A negative m (prediction) would correspond to a non-causal block G_2, containing $q^{-m} I$, in the control problem of Fig. 2. This is equivalent to forcing a delay $q^m I$, i.e. a computational delay, into the controller. (If $G_2 = q^{-m} I G_2'$, the block $q^{-m} I$ can be moved up to G_1 in Fig. 2, while its inverse, $q^m I$, is included in the controller.) With everything else being equal, the achievable performance would deteriorate as the prediction horizon/computation delay $-m$ increases.

(*d*) There are two ways of reducing the static feedforward control error: either G_1 or the output penalty G_5 should have high gain at low frequencies. Likewise, there are two ways to obtain an estimator with small error at low frequencies: either the frequency weighting G_1^{T} or the input model G_5^{T} should have high gain at low frequencies.

(*e*) The polynomial solutions to the two dual discrete-time problems, discussed in § 7, involve a spectral factorization, a diophantine equation and up to three coprime factorizations. (The same is true for the solutions to the corresponding continuous-time problems.) The transpositions used in going from one problem to the other explain why a *left* spectral factorization (10), where β appears to the left, is involved in the filtering solution, while a *right* spectral factorization (17) appears in the control solution. Also, note that while it is natural to start from a left MFD model (3) in the estimation problem, the dual control problem is expressed in right MFD form (13). See also Kučera (1991), where use is made of duality relations to investigate several other types of LQ problems, using the polynomial equations approach.

We have compared properties of the LQ (or H_2) solutions to the dual control and signal processing problems above. Very similar remarks apply to all criteria for which the duality holds, i.e. all norms for which $\|G^{\mathrm{T}}\| = \|G\|$. In particular, this applies to H_∞-optimal solutions.

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