System Identification for Deconvolution Filter Design and Equalization

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Abstract

The use of system identification for the design of a deconvolution estimator is addressed for discrete–time, linear, stable, time–invariant, single–input, single–output systems. A theoretical analysis on suboptimal design solutions in the presence of modeling errors is carried out for the linear deconvolution estimator (LDE) and for the decision feedback equalizer (DFE). A simple expression for the sensitivity of the performance of the Wiener deconvolution estimator with respect to unstructured perturbations of the optimal filter is obtained. The expression seems to be new. For the design of suboptimal filters, the criterion to be considered in order to minimize the loss of performance is obtained as a result. By means of computer simulations, it is shown that optimal filters of high order can effectively be approximated by suboptimal filters of low–order, with only a small performance degradation. A new principle for the MSE optimal design of DFEs is obtained, which leads to a novel method for suboptimal design. The problem of designing approximate DFEs is clarified and the role played by a constraint on the filter structure can be explained. A filter structure to be used for suboptimal design of DFEs is proposed. Strategies for approximate modeling to serve for LDE and DFE design are investigated and proposed. An extensive simulation study is carried out to evaluate the theoretical analysis and to draw some general conclusion on differences in performance of various methods for model estimation and filter design. The simulation study provides a basic point of reference for further experiments and indicates directions for further research.

Preface

This report is also available as

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Chapter 1

Design of Deconvolution Estimators from Observed Data

1.1 Introduction and Motivations

The use of *System Identification* for design purposes has recently received a renewed interest, [SpeAC92], [ScVHof92], [Ge93], [ModUnc94]. Mainly motivated by research activity on robust control, design problems based on observed data (data–based filter design) have been posed with more relaxed a–priori assumptions than in the past, and new, and challenging, aspects regarding the role of system identification as a tool for data–based filter design need to be investigated. In light of those aspects, the present study will analyse the potential offered by the use of system identification for designing an estimator algorithm, when very limited a–priori information is available, and only short input and (noisy) output data sequences are to be utilized.

We will investigate a classical signal processing problem, the design of a *deconvolution estimator*. The problem, further described in Example 1.1 below, consists of estimating an unknown input signal to a system, from noisy measurements of its output. A main motivation for studying this problem comes from digital communications, where receivers are adapted on–line by using short known data sequences. The analysis is geared toward the conventional, statistical system identification, and it will be conducted for discrete–time, linear, stable and time invariant, single–input, single–output systems, with output signal measured in the presence of additive measurement noise. Infinite impulse response structures for systems and models will be considered.

Data–based design will result in imperfect models. The study will therefore investigate ways to improve suboptimal solutions of the deconvolution problem, in the presence of modeling errors. Several methods for obtaining both models and design solutions are available in literature. It seems, however, that a general study on system identification for deconvolution filter design that takes modeling errors into account can not be found in the literature. Major aims of our study will thus be to draw attention on the subject, and to provide a reference for further studies.

The remaining part of the section provides an overview of the subject. The chapter then evolves with a summary of the literature background in the next section. In Section 1.3, the design problem will be stated, while in the following two sections, basic issues on the problem of data–based filter design will be discussed. The organization and the contribution of the study will be outlined in the final section of the chapter.

1.1.1 System Identification: a Tool for Data–Based Filter Design

Several engineering problems are solved by designing a filter for the processing of signals of interest. The design strategy usually follows a standard two–step procedure:

Modeling Stage: The situation under study is described by means of some formal language (e.g. mathematics, probability theory, computer programs, bonding graphs, etc.) leading to an abstract object called *model*, [LjuGla94]. The problem under study is also described by the use of such a formal language, leading to a *performance criterion* that measures the effectiveness of a solution to the problem.

Filter Design Stage: On the basis of the model and performance criterion, a solution to the problem is calculated in the form of a model of a physical device to be implemented subsequently. The design is usually performed by optimizing the performance criterion.

Example 1.1 A classical problem: the deconvolution problem. A common problem encountered in several fields is to obtain information on a quantity, say $u(t)$, when only a related quantity, say $y(t)$, can be observed. The argument "t" is a time index. A common strategy is to *model* the two quantities as realizations of mutually correlated stochastic processes. The problem then becomes a classical estimation problem. A more detailed description can be obtained by assuming that the signal $y(t)$ is the output of a system $\mathcal{S}$ driven by the input signal $u(t)$. The estimation problem is then called *deconvolution problem*, and it is depicted in Figure 1.1. The description of the system $\mathcal{S}$ and the probabilistic properties of the signal $u(t)$ form the *model*. With the model, the deconvolution problem can be solved by designing a filter, the deconvolution estimator $\mathcal{F}_u$, that is driven by the signal $y(t)$, and generates an estimate $\hat{u}(t)$ of the value $u(t)$. The performance criterion that measures the effectiveness of the design can, for instance, be chosen as the mean–squares of the estimation error

$$z(t) = u(t) - \hat{u}(t) \quad .$$

Since the signal $u(t)$ is, in general, unknown, the estimation error $z(t)$ can only be calculated during observation periods in which both $u(t)$ and $y(t)$ are available. In Figure 1.1, the two signals are measured at discrete time instants. The deconvolution problem is central in many engineering applications, as, for instance, communications, state estimation, fault detection, signal processing, seismology, ([Pro89], [Meh90], [StAh90], [NeHaEl92], [Cha84], [Men83], [Fox95]), and finds applications in other fields as well, including biomedicine, chemistry and economics, ([Baj83], [Com84]) ■

Figure 1.1: A general deconvolution problem, with signals measured at discrete time instants. The input signal $u(t)$ can be directly observed only during an observation period $t = 1, \ldots, T$. A distorted version of it, the output signal $y(t)$, is available. A deconvolution estimator $\mathcal{F}_u$ is to be designed for obtaining an estimate $\hat{u}(t)$ of unobserved input values $u(t)$.

During the past three decades, automatic methods for model building have received an ever growing interest. Models are then obtained on the basis of the observation of some variables that characterize the situation under study, in an automatic manner. The field of research is known as *System Identification*. Since the seminal survey on identification methods by Åström and Eykhoff in 1971, [ÅsEy71], several textbooks have embodied the theory in a unified framework. See, for instance, [Eyk74], [GoPay77], [Lju87], [SödSt89], [Boh91], [Reg94]. The use of system identification at the modeling stage allows the filter design to be automated to a large extent, and to be based on data observations.

During recent years, it has, however, been recognized that when obtaining models by system identification, the two stages mentioned above in the solution of design problems have remained somewhat disconnected, reducing the potential of application in several contexts. The interest on that issue was sparked off by another discipline, robust control, in the mid–eighties. Two major limiting factors were then brought to general attention:

- Models are usually obtained by utilizing a criterion different from the filter, or controller, performance. The process of modeling should instead be tailored to the final task at hand in order to optimize the total design.

- Since models are only approximate descriptions of the real data scenario, the model accuracy must be indicated. By taking the model uncertainty into account, methods for robust design can be applied, resulting in more reliable and effective performances. The description of the model uncertainty is heavily dependent on a–priori information, and algorithms for its estimation are not available for general situations.

For a general account on the above issues, see, for instance, [SpeAC92], [ModUnc94] and references therein.

In the system identification literature, the data generation mechanism, the "true system", is often assumed to be exactly described within a considered model class. Model errors have mostly been attributed to the presence of a stochastic disturbance affecting some measurements. Following the terminology used in [Reg94], this situation is referred as the *sufficient order case*. In the sufficient order case, model errors due to noisy data take the form of *variance errors*, [Lju87], [SödSt89]. If only a variance error[1] is present, the two factors mentioned above may not significantly limit the effectiveness of the design. Effective algorithms for estimation of the variance error are available for the sufficient order case. Furthermore, a model close to the optimal model for the design can often be obtained also when the identification criterion does not coincide with the filter performance criterion.

The sufficient order case will be rare in practice, as real systems are complex, possibly of infinite dimension, and frequently nonlinear. Also in cases when a sufficient order could be assumed by utilizing high order (linear) models, the complexity of a model will be limited by the number, and the quality, of the available data. It is often of interest to limit the duration of the identification experiment, for instance because of time variations of the environment. If the sufficient order case is not fulfilled, models will also be affected by *bias errors*, [WahLju86], [Lju87], [SödSt89]. A bias error accounts for the fact that the "true system" can not be described exactly within the considered class of models. This situation is called the *undermodeling case*. In the undermodeling case, the two limiting factors mentioned above may have to be taken into account in effective design solutions.

We will investigate these problems in relation to the deconvolution problem, introduced in Example 1.1, in the following chapters.

1.2 Literature Background

The literature on data–based filter design is vast. Different aspects of the topic have been studied in several fields, such as adaptive filtering, adaptive control, optimal and robust filter design and system identification. Below, we will summarize a few concepts that will be utilized repeatedly in the following chapters and mention the corresponding basic references. Subsequently, a brief overview of the relevant literature will be provided.

- The use of models as if they were the exact description of the data generation mechanism is denoted the *certainty equivalence principle*, [ÅsWit89]. Design methods based on the certainty equivalence principle are called *nominal designs*, or nominal solutions. We will consider the use of Wiener filters, designed as in [AhSt89], and

[1] If the system that generated the data is time–invariant, experiments using the same input sequence will generate different output sequences due to different realizations of the measurement noise. Different models would therefore be obtained in repeated identification experiments. The variance error can be reduced by increasing the duration of the identification experiment, or by repetitions of experiments.

(MSE optimal) decision feedback equalizers, designed as in [StAh90].

- *Robust design methods* take also a description of the model uncertainty into account. We will consider the use of a robust method for Wiener filter design presented in [StAh93]. The use of other robust design methods is left for further research on the subject.

- Nominal and robust design methods are referred as *model–based design*, since in the literature a model is usually assumed to be given, [AndMo79], [Cad87], [Söd94]. An alternative is the *direct adaptation* of the filter to the available data, [Hay86]. Both methodologies will be investigated in our study.

- Our standard references on system identification are [Lju87] and [SödSt89].

- When estimating models from observed data, we introduced, in the previous section, the distinction between two cases: the *sufficient order case* and the *undermodeling case*. The terminology follows the one used in [Reg94]. The undermodeling case will be the case treated in our study.

- In [Kai74], two cases of filtering theory are distinguished: the *probabilistic* theory and the *statistical* theory. In the probabilistic theory, a probabilistic description of the data scenario is assumed to be known (and a model–based design is considered), while in the statistical theory it has to be inferred from data observation. According to this distinction, a statistical theory of filter design is considered in our study.

1.2.1 A Brief Literature Overview

In classical filter design, filters are seen as circuits or systems with frequency selective behavior that operate on continuous time signals, such as, for instance, voltages and currents. In general, the design is based on specifications given in the frequency domain in the form of filter bandwidth, roll–off, attenuation, phase distortion, etc. Such specifications are inferred mainly from physical principles, heuristics and engineering insight. See, for instance, [AndMo79] for an historical overview on filter design.

Modern filtering theory was brought forward by two major developments: the advent of digital filtering, [AndMo79], [Cad87], [OppSc89], [Söd94], and the application of statistical ideas to filtering problems, [Wie49], [Kol41], [BoSha50]. In digital filtering, the filter is represented by an algorithm that processes data in the form of numbers. The algorithm is specified by a *structure* and a *set of parameters*, that determine the data–flow. The filter design thus corresponds to selecting an opportune structure and appropriate parameter values. If the algorithm is linear, the filter is uniquely specified by its impulse response, [OppSc89]. The statistical approaches to filtering postulate that signals have certain statistical properties, [AndMo79], [Cad87]. This, in turn, leads to the assumption that the process of data generation can be described by an algorithm, i.e by *a model*, [Söd94]. A significant aspect of the statistical approach is that it simplifies the definition of a measure of performance of a given filter. After expressing the performance measure in terms of the filter impulse response and the statistical properties of the signals (the model), an optimal filter can be designed. The literature is rich of (optimal) design methods based on assuming exact knowledge of the data generation mechanism. In our study, we will consider the well known Wiener filter, [AndMo79], [Cad87], [AhSt89], [Söd94], and the decision feedback equalizer,[BePa79], [StAh90], for the solution of a deconvolution problem defined in statistical terms in the next section.

Exact modeling may often not be attained in practice. One may then still use the optimal solutions based on an incorrect model, as if the model were exact, on the basis of the *certainty equivalence principle*. We referred to those solutions as the *nominal solutions*. With incorrect models, the performance of nominal solutions may be significantly degraded, as compared to the nominal (theoretical) performance. The reason is that the performance of a filter that is calculated from a certain model assumption can be sensitive to the model assumption itself. To address this problem, the use of *robust solutions*, where the design is carried out on the basis of some description of the modeling errors, have been proposed in the literature. Pioneering studies on robust filtering appeared in the late sixties, [DapHu69], [LePe72], [DapHu72]. For an account on a general theory of robust Wiener filters, see, for instance, [Po80], [VaPo84]. For the use of robust techniques in signal processing problems, see [KasPo85] and references therein. Robust solutions for the design of decision feedback equalizers have also appeared in the literature, see [LiAhSt93], [StAhLi93]. In Section 1.5, we will further comment upon the use of robust solutions for deconvolution filter design.

Both nominal and robust designs require models of the data scenario, either in the form of a nominal model, or, in general, of a nominal model plus some description of the model uncertainty. In the literature on filter design, the model is usually assumed to be given. We referred to those design methods as *model–based designs*. The problem of obtaining *models* from the observation of data sequences is studied in the field of System Identification. See the previous section. The problem of estimating models in forms that are suitable for robust designs has received attention only recently. Interest has been mainly directed toward methods for robust controller design, see [SpeAC92] and [ModUnc94] for an overview of the subject. We will further comment upon such an aspect in Section 1.5.

The problem of designing *filters* on the basis of data observations has traditionally been studied within the field of Adaptive Filtering, see, for instance, [Hay86], [TrJoLa87]. Particular attention has there been given to *direct adaptation* of finite impulse response (FIR) filters in time–invariant or time–varying environments. The main interest is on the adaptive algorithm for parameter estimation. The filter structure is determined from considerations other than optimality with respect to the performance. For instance, FIR structures have been often preferred over infinite impulse response (IIR) structures, due to simplicity of implementation, and the avoidance of stability problems, ([TrJoLa87], Chapter 7). The use of IIR structures has, however, also been considered, [John84], [Reg94].

In the field of Adaptive Control, the problem of designing controllers from data observations is considered, [ÅsWit89]. Crucial a–priori information on the system to be controlled is often assumed to be available. Particular emphasis is given to the adaptive algorithm and to the global stability of the design. Solutions based on the tailoring of system identification to the task at hand, through an iterated design process, have recently been proposed in the literature. See, for instance, [ScVHof92], [Ge93], [ModUnc94]. Such methodologies can be regarded as (very) slow adaptive control schemes.

In 1974, in an overview on thirty years of linear filtering theory, Thomas Kailath wrote, [Kai74],

> ... The development of a *statistical* theory of filtering will introduce several new dimensions, although it might be noted that the lack of a complete statistical theory does not seem to have significantly limited the successful use of the ideas of the *probabilistic* theory.

The statistical theory is here contrasted to the probabilistic theory. The distinction is on whether the probabilistic model is assumed to be known, or whether it has to be inferred from data observation. According to the distinction above, a statistical theory of filter design will be considered in this study.

1.3 The Design Problem

In this section, the deconvolution problem will be outlined and the considered design problem will be stated. Data are assumed to be accurately described by the following system.

Definition 1.1 The True System. Data are generated by the discrete–time, linear, time–invariant and stable system

$$\begin{aligned} y(t) &= G(q^{-1})u(t) + v(t) \\ v(t) &= H(q^{-1})e(t) \\ u(t) &= H_u(q^{-1})w(t) \end{aligned} \tag{1.3.1}$$

$$\lambda_w = \mathrm{E}w^2(t) \quad ; \quad \lambda_e = \mathrm{E}e^2(t) \quad ; \quad \lambda_u = \mathrm{E}u^2(t)$$

where the input signal $u(t)$ and the measurement noise $v(t)$ are stationary, real valued, zero–mean stochastic processes, represented by innovation models. The transfer functions $H_u(q^{-1})$ and $H(q^{-1})$ will thus be stable and minimum phase, with the first impulse response coefficient equal to one. The input and the noise are mutually uncorrelated. The input signal $u(t)$ may take values on a finite alphabet $\mathcal{A}_u$, with equally likely symbols. In that case, the input will be assumed to be white. The stable transfer functions in (1.3.1) are expressed by polynomials in the backward shift operator q^{-1}:

$$\begin{aligned} G(q^{-1}) &= \frac{B(q^{-1})}{A(q^{-1})} = \sum_{k=0}^{\infty} g_k q^{-k} \\ H(q^{-1}) &= \frac{M(q^{-1})}{N(q^{-1})} = 1 + \sum_{k=1}^{\infty} h_k q^{-k} \\ H_u(q^{-1}) &= \frac{C(q^{-1})}{D(q^{-1})} \quad . \end{aligned}$$

The following variance ratios will also be utilized repeatedly:

$$\rho \triangleq \lambda_e/\lambda_w \quad ; \quad \rho_u \triangleq \lambda_u/\lambda_w \quad .$$

The system (1.3.1) is represented by the quadruple

$$S = \{G, H, \rho, H_u\} \quad .$$

The subscript "T" will be used to distinguish polynomials and other parameters related to the system from those related to the model when necessary[2] ■

Remark 1.1 Notations for polynomials. A polynomial $P(q^{-1})$, of degree np, is expressed as

$$P(q^{-1}) = p_0 + p_1 q^{-1} + p_2 q^{-2} + \ldots + p_{np} q^{-np}$$

where the coefficients $\{p_i\}$ are real valued. For the polynomial $P(q^{-1})$, the *conjugate polynomial* is defined as

$$P_*(q) \triangleq p_0 + p_1 q + p_2 q^2 + \ldots + p_{np} q^{np} \quad .$$

Corresponding notations are used for other polynomials ■

Denote the past values of the output signal $y(t)$ up to the time instant t by

$$Y_t \quad .$$

For the system S in (1.3.1), the deconvolution problem is defined as follows. The problem is depicted in Figure 1.2.

Definition 1.2 The Deconvolution Problem. An estimate $\hat{u}(t-m)$ of $u(t-m)$ is sought, based on Y_t. The time lag m may denote filtering ($m = 0$), fixed lag smoothing ($m > 0$) or prediction ($m < 0$). The case of prediction will however not be considered. The input estimate will thus be obtained by

$$\hat{u}(t-m) = \mathcal{F}_u(Y_t) \tag{1.3.2}$$

for a causal algorithm $\mathcal{F}_u$, with the resulting estimation error

$$z(t) = u(t-m) - \mathcal{F}_u(Y_t) \quad . \tag{1.3.3}$$

The performance of an algorithm $\mathcal{F}_u$ in (1.3.2) is measured by the mean–square criterion[3]

$$V(S, \mathcal{F}_u) = \mathrm{E} z^2(t) \quad . \tag{1.3.4}$$

The objective is to obtain the algorithm that minimizes the mean–square criterion (1.3.4) over a given subset of causal algorithms ■

[2]The basic literature references on system identification and filter design methods, see Section 1.2, differ in their standard of notations. The notations we use mainly corresponds to those used in system identification. Modifications have been introduced on the notations for polynomials, extensively used in the following chapters. This choice has been made for a better comparison to the original references on filter design.

[3]The fact that the algorithm $\mathcal{F}_u$ is applied on data generated by the system S is stressed by the notation $V(S, \mathcal{F}_u)$.

Figure 1.2: The considered deconvolution problem, where m is the smoothing lag. The algorithm $\mathcal{F}_u$ is to be designed so that the variance of the estimation error $z(t)$ is minimized. The "true system" is unknown and only input and output data in the interval $t = 1, \ldots, N_d$ are available to the designer. Two estimation schemes will be considered, namely linear deconvolution estimators and decision feedback equalizers.

Deconvolution problems have been studied extensively in the literature, and several deconvolution algorithms have been proposed. Performance criteria different from (1.3.4) have also been considered. For a general account on the deconvolution problem, see, for instance, [Men83], [Com84], [Moir*et*87], [StAh93], [Fox95].

We will consider the following estimator schemes:

- Linear Deconvolution Estimators,
- Decision Feedback Equalizers (DFEs).

The *linear deconvolution estimator* consists of a linear and stable filter. If the input takes values in a finite alphabet, a nonlinear decision block, where decisions are based on threshold levels, is added in cascade to the filter. In that case, the resulting scheme is that of the *linear equalizer*. For a given filter $F(q^{-1})$, the input estimate, before a possible decision block, is obtained by

$$\hat{u}(t-m) = F(q^{-1})y(t) \quad . \tag{1.3.5}$$

The objective is to solve the deconvolution problem over the set of linear and stable filters. The solution is given by the Wiener and the (asymptotic) Kalman filters, see [AndMo79], [Men83], [Moir86], [AhSt89]. Linear deconvolution estimators will be considered in Chapter 2.

The *decision feedback equalizer* (DFE), [BePa79], [StAh90], is a simple symbol–by–symbol detector of inputs that take values in a finite alphabet. A well designed DFE will, in general, provide a much higher performance than a linear equalizer, in particular for channels with zeros close to the unit circle. Previous symbol estimates are used to reduce the intersymbol interference which affects the received symbols. The DFE consists of two linear filters, namely the feedforward filter $F_f(q^{-1})$ and feedback filter $F_b(q^{-1})$, and a nonlinear decision block where decisions are based on threshold levels. The input estimate, before the decision block, is formed as

$$\hat{u}(t-m) = F_f(q^{-1})y(t) - F_b(q^{-1})\bar{u}(t-m-1) \tag{1.3.6}$$

where $\bar{u}(t-m-1)$ are symbols previously estimated. The objective is to solve the deconvolution problem over the set of linear and stable filters $F_f(q^{-1})$ and $F_b(q^{-1})$. Decision feedback equalizers will be considered in Chapter 3.

The design problem addressed in this study is stated as follows.

Definition 1.3 The Design Problem. Data are assumed to be generated by the system (1.3.1). The system is unknown, and the input and the output sequences are observed during a *training*, or identification period

$$t = 1, \ldots, N_d \quad .$$

After this period, the input is unknown, and will have to be estimated by the designed algorithm. The following a–priori information on the system (1.3.1) is available:

- the input and output signals have zero mean,
- the input spectrum is known, and if the input signal takes values in a finite alphabet, the alphabet and the probability of the symbols are known,
- the channel $G(q^{-1})$ is time–invariant, stable, and can be accurately described by a rational transfer function (of possibly high order),
- the channel has no pure time delay (that is to say that a possible pure time delay is known),
- the noise can be accurately described by a zero–mean stationary stochastic process with rational spectrum,
- the input and the noise are mutually uncorrelated.

The problem is then to design either a linear deconvolution estimator or a decision feedback equalizer as the solution of the deconvolution problem for the system (1.3.1). Emphasis will be given to design methods based on parametric models obtained by system identification ■

The main purpose of the study is to investigate the effect of undermodeling on the attainable performance. The a–priori assumptions have therefore been selected to limit other causes of modeling errors. The problem of *blind equalization*, $N_d = 0$, [Hay94], will not be considered.

If the number of data N_d is large enough, the sufficient order case may, in principle, be assumed, and a rather detailed model can be provided as the basis for some model–based designs. It is thus of interest to consider situations with only short data sequences available for the design. The number of parameters that can be utilized in a successful model will then be limited. Models obtained by system identification will be effected by both *variance errors* and *bias errors*, [Lju87], [SödSt89]. The effect that model errors will have on the performance of the resulting estimator is not clear, and it will be investigated in the following chapters.

1.4 Direct and Indirect Methods

The use of the standard two–stage procedure for the solution of design problems introduced in Section 1.1 is not the only possible strategy when data are available. In adaptive control theory, where the problem is to design a controller on the basis of data observations, two design approaches are distinguished, [ÅsWit89]:

- indirect methods,
- direct methods.

Indirect methods correspond to the standard two–stage procedure. With direct methods, a design solution is instead directly fit to the available data, without explicit modeling. The same two classes of methods are found in signal processing and digital communications, where their relative advantages and disadvantages for filter design are currently debated. In this section we will illustrate the two approaches. Basic considerations on their use for the deconvolution design problem will also be given.

Direct Methods. Consider the design of a linear deconvolution estimator. For a given linear and stable filter $F(q^{-1})$, the input estimate is obtained by (1.3.5), with the resulting estimation error, refer to Figure 1.2, given by

$$z(t) = u(t-m) - F(q^{-1})y(t) \quad . \tag{1.4.1}$$

The objective of the design is to obtain the filter $F(q^{-1})$ in (1.3.5) that minimizes the mean–square criterion

$$V(S_T, F) = \mathrm{E}z^2(t) \tag{1.4.2}$$

over the set of all causal, stable and linear filters. The criterion (1.4.2) is not available, and only the sample performance during the observation period can be computed as

$$V_{dir}(F) = \frac{1}{N_d - m} \sum_{t=m+1}^{N_d} z^2(t) \quad . \tag{1.4.3}$$

Assume that a filter class $\mathcal{F}$ has been selected. Then a filter $F_{dir}(q^{-1})$ is designed with a direct method, by solving

$$F_{dir}(q^{-1}) = \arg\min_{\mathcal{F}} V_{dir}(F) \quad . \tag{1.4.4}$$

In principle, the minimization problem in (1.4.4) can be solved, for instance by the use of gradient methods. The solution may, however, converge to local, but not global, minima of the criterion (1.4.3). Asymptotically in the number of training data, the sample performance (1.4.3) will coincide with the deconvolution performance (1.4.2), because of the ergodic theorem, [SödSt89]. Therefore, in principle the direct method will asymptotically provide the optimal design within a given filter class $\mathcal{F}$.

Denote the optimal filter in the filter class $\mathcal{F}$ by $F_{opt}(q^{-1})$. By rewriting the expression (1.4.1) as

$$u(t-m) = F(q^{-1})y(t) + z(t) \tag{1.4.5}$$

$$W_{id}(F) \triangleq V_{dir}(F) \ , \tag{1.4.6}$$

the design of a linear deconvolution filter with a direct method is seen to correspond to identifying the transfer function $F_{opt}(q^{-1})$ with an output–error method, [SödSt89], based on the model equation (1.4.5) with the signal $u(t-m)$ as output and the signal $y(t)$ as input, and the sample performance (1.4.3) as the loss–function $W_{id}(F)$ of the identification algorithm.

The filters in the decision feedback equalizer (DFE) are adjusted in a similar way. During the training period, the known input sequence can be utilized in the feedback path of the equalizer instead of decisioned data $\bar{u}(t)$. During that period, the input estimate (1.3.6) can thus be expressed as

$$\hat{u}(t-m) = F_f(q^{-1})y(t) - F_b(q^{-1})u(t-m-1) \tag{1.4.7}$$

where $F_f(q^{-1})$ is the feedforward filter and $F_b(q^{-1})$ is the feedback filter. The filters can therefore be estimated from the model equation

$$u(t-m) = F_f(q^{-1})y(t) - F_b(q^{-1})u(t-m-1) + z(t) \tag{1.4.8}$$

$$W_{id}(F_f, F_b) = \frac{1}{N_d} \sum_{t=m+1}^{N_d} z^2(t) \ . \tag{1.4.9}$$

Note that the estimation of the DFE filters corresponds to an identification problem with one output, $u(t-m)$, and two inputs, $y(t)$ and $u(t-m-1)$. It can be reduced to a single–input problem, by rewriting the model equation (1.4.8) as

$$\left[1 + q^{-1}F_b(q^{-1})\right] u(t-m) = F_f(q^{-1})y(t) + z(t) \ . \tag{1.4.10}$$

The same asymptotic considerations as for the linear deconvolution filter apply for the DFE. The direct method is the design approach most commonly utilized in communications technology, where FIR structures are usually implemented. Direct adaptation of linear FIR equalizers was first proposed by Robert Lucky at Bell Lab in 1965, [Luc65]. Direct adaptation of DFEs, with FIR feedback and feedforward filters, was proposed by George *et al* in 1971, [Ge*etal*71], and John Proakis in 1975, [Pro75] ■

Indirect Methods. With indirect methods, first a model

$$S = \{G, H, \rho\} \tag{1.4.11}$$

is obtained for the system S_T by using an identification algorithm. We will consider the use of prediction based algorithms, see Section 4.3, in which the model S in (1.4.11) is obtained by minimizing a loss–function, denoted by

$$W_{id}(S) \quad . \tag{1.4.12}$$

As a second step, a model–based design is then utilized, and the design will result in the deconvolution estimator denoted by

$$F = \mathcal{D}(S) \tag{1.4.13}$$

where F represents either a linear deconvolution estimator or a decision feedback equalizer, and where $\mathcal{D}$ represents the design equations of the considered method. Note that the filter class $\mathcal{F}$ will be determined by the choice of model class $\mathcal{S}$, and by the design law represented by (1.4.13). Asymptotically in the number of training data, the attained sample performance $V_{ind}(F)$ will coincide with the deconvolution performance

$$V_{ind}(F) = \frac{1}{N_d} \sum_{t=m+1}^{N_d} z^2(t) \quad \overset{N_d \to \infty}{\longrightarrow} \quad V(S_T, \mathcal{D}(S)) \quad . \tag{1.4.14}$$

In the sufficient order case, the asymptotic sample performance (1.4.14) is minimized by the correct model description of the system S_T, which can be obtained by a consistent identification algorithm[4]. It that case, the optimal design over the set of causal, linear and stable estimators will also be attained. In the undermodeling case, it is very difficult, if not impossible, to minimize the asymptotic sample performance (1.4.14) with respect to the model S, due to the nonlinear transformation of the parameters introduced by the design equations. This problem, mentioned in Section 1.1 as one of the two limiting factors to the use of system identification for design purposes, will be central in our study, and strategies for how to appropriately select approximate models for indirect methods will be studied and proposed in the following chapters ■

The relative merits of direct and indirect methods. Asymptotic considerations constitute only one aspect of data–based design, and several other aspects have to be taken into account. For instance, direct methods are simpler to use than indirect methods, since there is no need for design equations.

As previously mentioned, models, and filters designed with direct methods, will be affected by a variance error. The variance error will depend on how the noise during the training sequence affects the loss–function of the identification algorithm. This issue will be considered in Chapter 4. It will be argued, and confirmed by simulation studies in Chapter 5, that direct methods are more sensitive; the resulting performance is affected more by variance errors, than with indirect methods. Since, in general, the variance error increases with the number of estimated parameters, the sensitivity will limit the number of parameters that can be estimated in a meaningful design. For instance, in the case of multiple antenna receivers, outputs from several channels are utilized for input estimation, [LiAhSt93]. In such a case, the use of direct methods does not seem feasible because of the high number of parameters that have to be estimated (one filter has to be

[4]A *consistent* estimate is an estimate that converges, asymptotically, to the "true value" of the quantity under estimation, [SödSt89].

designed for every channel). Indirect methods here result in a much higher performance for short training data set, see [LiAhSt93], [LiAhSt95].

With both direct and indirect methods, the design is obtained by solving a minimization problem, and the nature of the minimization will be important. For instance, one but not the other approach may result in a quadratic performance surface in the variables, providing a unique global minimum that can be easily calculated. This is the case when linear deconvolution filters are designed for FIR channels with indirect methods, while direct optimization of the corresponding optimal filter structure (a rational filter) will result in a nonconvex minimization problem, with possible local minima.

A situation that favors indirect methods is when filters have to be designed in time–varying environments. In [Lind95], it is shown that tracking of system parameters is significantly simpler than tracking of filter parameters, since slow variation in the system parameters may cause abrupt changes in the parameters of the filter.

The selection of a model or a filter class is a difficult problem if only qualitative information on the data scenario is available. The problem is related to that of model validation and structure selection, [Lju87], [SödSt89], that is known to be complex and to require rather large sets of data. A brief overview on the problem will be given in Section 4.5. When designing linear deconvolution filters, validation of a model and validation of a filter structure are, in principle, equivalent problems. When designing decision feedback equalizers, the filter structure selection in a direct method is much more cumbersome, since the interaction of the feedforward and feedback filters is obscure without knowledge of a signal model ■

In summary, it is not clear which of the two design approaches should be preferred over the other in general, so both strategies have to be considered. Comparison of direct and indirect methods is a basic issue that will be considered in parallel with the analysis of indirect methods. It will also be central in the experimental investigations of Chapter 5, based on simulated data.

1.5 The Use of Robust Designs

It has already been mentioned in the previous sections that nominal designs, based on incorrect models, may result in poor performances when applied on a real data scenario. The design performance can be improved by using *robust designs*, see Section 1.2. The use of robust designs when (parametric) models are obtained by system identification may however be complicated. In particular, two basic and general problems arise. These problems are introduced in this section, and will be further considered in Section 2.5, related to the method for robust Wiener filter design presented in [StAh93], denoted the *cautious Wiener filter.*

The first problem related to the use of robust methods is the difficulty of providing an uncertainty description in the case of undermodeling. This issue was mentioned in Section 1.1 as one of the two limiting factors to the use of system identification for design purposes. The problem of estimating models in forms that are suitable for robust designs has received (growing) attention only during the past few years. As stated in [HjaPhD93], "this topic is still in its infancy", and simple algorithms for uncertainty estimation are not available in the literature. Several methods have been proposed, but their use may have limitations when applied to general situations. For instance, the methods may require particular model structures, as the method based on the stochastic embedding philosophy, [GoGeNi92], particular input signals, as in the mixed deterministic–probabilistic approach of [deVri94], or utilize frameworks different from the stochastic noise assumption, as in the identification framework based on an unknown–but–bounded noise either in the time domain, [MiVi91], or in the frequency domain, [Par91].

The second problem is that two cases have to be distinguished when designing a robust filter for time–invariant systems, [StAh93]:

Case 1: One filter is to be applied on a whole class of "true systems"

$$\mathcal{S}_T = \{S_{T1}, \ldots, S_{Tn}, \ldots\} \quad .$$

Case 2: For an unknown, and unique, "true system" S_T, a set of different models

$$\mathcal{S} = \{S_1, \ldots, S_n, \ldots\}$$

is available. From every model S_i, a filter $F_i(q^{-1})$ is designed with a model–based design method. This leads to a set of filters

$$\mathcal{F} = \left\{F_1(q^{-1}), \ldots, F_n(q^{-1}), \ldots\right\}$$

to be applied to the system S_T.

Case 1 corresponds to the design scenario on which robust methods apply. Robust methods differ in how the class of systems $\mathcal{S}_T$ is characterized and what performance criterion is considered. For instance, the class of systems may be given by a nominal model with bounds on the parameters, [Xi*etal*94], [BoCoNi94], by integral constraints on the noise spectrum, [VaPo83], by a probability measure, [StAh93], [ChCh94], etc. The robust design may focus on worst-case filtering performance over the class of systems, [VaPo83], [ChCh94], or average performances, [StAh93], [Öhrn95]. Criteria that take other aspects besides the filtering performance into account have also been considered, [ChPe94]. In general terms, the set $\mathcal{S}_T$ represents the a–priori information available on an unknown system under study, which is assumed to belong to the set. The framework of Case 1 can also be used to describe the application of a time–invariant filter to a (slowly) time–varying system, see, e.g. [LiAhSt93], [BoCoNi94], or to a non–linear system, where the set $\mathcal{S}_T$ represents a set of different operating conditions.

Case 2 corresponds to the case when models are obtained by system identification. Due to the variance error, different models would be obtained in repeated identification experiments.

Apparently, Case 2 differs from Case 1. It is not clear whether solutions for design problems belonging to Case 1 are sensible solutions also for design problems belonging to Case 2. In principle, a situation similar to Case 1 can be inferred from Case 2 in the sufficient order case. The true system S_T is known to belong to the model class, and the uncertainty is represented by a variance error, which can be efficiently estimated, [SödSt89]. The class of systems $\mathcal{S}_T$ can then, for instance, be modeled by a set of transfer functions with parameters described as stochastic variables with known first and second order moments. An experimental investigation was conducted in [StAh93] for FIR models with parametric uncertainty. In that case, the results indicate that robust filters designed for Case 1 can be applicable also in Case 2. In the undermodeling case, the recovery of a Case 1 design problem is more complicated, for instance, because of the problem of estimating the model uncertainty mentioned above. For design problems belonging to Case 2, the set $\mathcal{S}_T$ defined for design problems belonging to Case 1 may rather be seen as a "tuning knob", to be adjusted on the data until a satisfactory design performance is attained, [StAh93].

The above issues will be investigated by means of computer simulations in Chapter 5, related to the use of the cautious Wiener filter for the solution of the design problem stated in Section 1.3.

1.6 Outline and Reader's Guide

A theoretical analysis on suboptimal solutions of the deconvolution problem in the presence of modeling errors is carried out for the linear deconvolution estimator, in Chapter 2, and for the decision feedback equalizer, in Chapter 3. In Chapter 4, some issues regarding the use of system identification are discussed. In particular, the sample behavior of the two basic design approaches, direct and indirect methods, is analysed. In Chapter 5, the theoretical analysis of the previous chapters is then evaluated and compared by means of computer simulations. The Appendices A and B contain the proofs of theorems and lemma in Chapter 2 and 3, respectively. The main conclusions drawn from the study are briefly summarized in a separate section, following Chapter 5.

The organization of the study is depicted in Figure 1.3.

The chapters are separately readable, and the contribution of each chapter is summarized in its introduction. In Chapters 2 and 3, a brief overview and historical account on the investigated subject is also given for the convenience of the reader. With the same aim, a certain degree of redundancy, as for instance notations repeated in each chapter, has also been maintained.

A significant conclusion that will emerge from the study will be that the two limiting factors to the use of system identification for design purposes, mentioned in Section 1.1, will turn out *not* to constitute a serious limit for a successful design of deconvolution estimators for time–invariant, stable and linear systems. This conclusion is drawn mainly from the simulation study of Chapter 5, conducted with several systems of (relatively) high order (12 zeros and 12 poles) with various complexity. The design was carried out in situations with short (50–300 samples), and noisy (SNR = 13dB) data sequences. More than the optimization of the modeling stage and the use of complicated design methods, refinements of the identification algorithms, especially those for model validation, to cope

Figure 1.3: System identification for deconvolution filter design and equalization. Organization of the study and reader's guide.

with situations with short data sets seem to represent the crucial factor for a successful use of (nominal) indirect filter designs.

Below follows an outline of each chapter.

Chapter 2: The Linear Deconvolution Estimator. The optimal design of Wiener filters using polynomial equations is summarized in Section 2.2, where some additional insights and simplifications are also presented. The main aim is however to investigate and improve suboptimal solutions, in the presence of modeling errors. In order to assess the degradation of the optimal performance and to determine sensible strategies for suboptimal design, a sensitivity analysis of the (optimal) performance with respect to unstructured perturbations of the optimal filter is carried out in Section 2.3. It is shown that the performance is determined by the relative (percentage) perturbation in the frequency domain of the optimal filter, via a simple integral relation. The expression seems to be new. For the design of suboptimal filters, the criterion to be considered in order to minimize the loss of performance is obtained as a result. The sensitivity of the optimal design can also be easily assessed. By means of computer simulations, it is shown that optimal filters of high order can effectively be approximated by suboptimal filters of low–order, with only a small performance degradation. In Section 2.4, the results of Section 2.3 will serve as the basis for studying how to appropriately select approximate models to serve for the design of suboptimal filters. In the final section, the method for robust Wiener filter design presented in [StAh93] is summarized, and its use discussed.

Chapter 3: The Decision Feedback Equalizer. The MSE optimal DFE derived in [StAh90] is reconsidered in Section 3.2. It is shown that the estimation mechanism of an optimal design can be seen as being based on two separated stages. In the first stage, optimal linear predictions of the output process $y(t)$ are calculated from past input–output values up to time $t-m-1$, where m is the smoothing lag. Then, in the second stage, the estimate of $u(t-m)$ is obtained as the optimal linear mean square estimate based on the corresponding prediction errors. This result is important for two reasons. First, a new principle for the optimal design of DFEs is obtained, which leads to a novel method for suboptimal design. Second, the problem of designing approximate DFE schemes is clarified and the role played by a constraint on the filter structure can be explained. The above issues are analyzed in Section 3.3, where a filter structure to be used for suboptimal design of DFEs is also proposed. In the final section, strategies for approximate modeling to serve for DFE design are investigated. An optimal strategy is found in the case of filtering. The method can also be useful for the design of DFEs for smoothing, as will be illustrated by means of two examples in the final section.

Chapter 4: Some Issues on System Identification. The analysis in Chapters 2 and 3 is based on asymptotic considerations. The sample behavior of the two basic design approaches, direct and indirect methods, is analysed in Sections 4.2 and 4.3, respectively. On the basis of a qualitative analysis, it is argued that direct methods result in designs more sensitive to the noise realization during the identification experiment than those attained by indirect methods. By means of a computer simulation, it is shown that direct methods can in fact result in designs with a very poor performance. The conclusions inferred from the analysis of the sample behavior will be confirmed by the simulations studies in Chapter 5. In Section 4.3, general considerations concerning the estimation of models for indirect methods are also given. In Section 4.4, we investigate the effect of unknown initial conditions on the filter design in situations with short data sequences, while in Section 4.5, a brief overview of the problem of model validation or structure selection is given.

Chapter 5: Experiments on Simulated Data. A set of experiments is carried out, with three main purposes:

- to evaluate the theoretical analysis on simulated data, generated to cover a wide variety of test situations,
- to possibly draw some general conclusion on differences in performance between methods for model estimation and filter design,
- to obtain a basic point of reference for further experiments and to investigate directions for further research.

Two problems are studied, namely, the design of a filtering estimator ($m = 0$) and that of a smoother with $m = 4$. Emphasis is given to situations with short data sequences, with only 50 samples of input and output data, with signal to noise ratio SNR = 13dB, available for the design. As a general conclusion, it seems that models obtained by system identification are models "good enough" for the design of deconvolution filters with nominal indirect methods, also when the identification experiment is performed with relatively few, and noisy, data, and the underlying system is complex. The use of an indirect nominal design turns out to result in performances comparable to those attained with a robust indirect method. The design of IIR DFEs is more difficult, and, with short data sequences available for the design, the resulting performance can deteriorate markedly. Good model identification, and in particular good model validation, seems to be the crucial factor for a successful use of indirect methods. It is definitely more important than the introduction of more complicated methods in the filter design stage.

Chapter 2

The Linear Deconvolution Estimator

2.1 Introduction

In this chapter, the use of a *linear filter* as the solution of the deconvolution problem described in Section 1.3 will be analyzed. The optimal solution is, in that case, given by the *Wiener* and the (asymptotic) *Kalman* filters, [LaKa71], [FiKu75], [AndMo79], [Men83], [Moir86], [AhSt89], [AhSt91]. The estimation scheme is depicted in Figure 2.1.

The optimal design of Wiener filters using polynomial equations is summarized in Section 2.2, where some additional insights and simplifications are also presented. The main aim is however to investigate and improve suboptimal solutions, in the presence of modeling errors. In order to assess the degradation of the optimal performance and to determine sensible strategies for suboptimal design, a sensitivity analysis of the (optimal) performance with respect to unstructured perturbations of the optimal filter is carried out in Section 2.3. It is shown that the performance is determined by the relative (percentage) perturbation in the frequency domain of the optimal filter, via a simple integral relation. The expression seems to be new. For the design of suboptimal filters, the criterion to be considered in order to minimize the loss of performance is obtained as a result. The sensitivity of the optimal design can also be easily assessed. By means of computer simulations, it is shown that optimal filters of high order can effectively be approximated by suboptimal filters of low–order, with only a small performance degradation. In Section 2.4, the results of Section 2.3 will serve as the basis for studying how to appropriately select approximate models to serve for the design of suboptimal filters. In the final section, the method for robust Wiener filter design presented in [StAh93] is summarized, and its use discussed. The strategies for the design of suboptimal deconvolution filters will be studied by computer simulations in Chapter 5.

Figure 2.1: The linear deconvolution estimator, where m is the smoothing lag. The transfer function $F(q^{-1})$ is to be designed so that the variance of the estimation error $z(t)$ is minimized. If the input signal $u(t)$ takes values on a finite alphabet $\mathcal{A}_u$, a nonlinear decision block, where decisions are based on threshold levels, is added in cascade to the linear deconvolution filter. In that case, the resulting scheme is known as a *linear equalizer*.

2.1.1 An Overview

The Wiener filter has a long history that dates back to the original work by Wiener and Kolmogorov in the forties, [Wie49], [Kol41], [BoSha50]. Since then, a vast literature has appeared on the subject, see, for instance, [vTre68], [AndMo79], [Cad87], [Söd94] and the references therein. The problem of designing linear filters from observed data has been a major topic of research within the field of adaptive filtering. Particular attention has there been given to direct adaptation of FIR filters in time–varying environments, [Hay86], [TrJoLa87]. See also Section 1.2.

In this chapter we shall investigate indirect methods for Wiener filter design based on parametric models obtained by system identification [1]. Models obtained in this way are affected by *variance errors* and *bias errors*, [Lju87], [SödSt89]. See Section 1.1. Optimal deconvolution filters can therefore not be designed, and only suboptimal filters can be obtained. Under such circumstances, *robust* solutions, where the design is carried out on the basis of some description of the model mismatch, has been proposed.

The use of a robust design may however be complicated, and it can be difficult to obtain an uncertainty description suitable for the design method. See Sections 1.5 and 2.5. The use of the *certainty equivalence principle* is an obvious and perhaps reasonable alternative. The available model is then utilized as if it were correct, and the deconvolution filter is thus designed as a Wiener filter for the model. In Section 1.2, this strategy was referred as the *nominal design* solution. It is not clear under what circumstances a sensible design can be obtained by using nominal solutions also in the undermodeled case, or how models should be derived in order to optimize the total design. We will address the above questions in the following sections.

A nominal design based on an incorrect model will result in a perturbation of the optimal

[1] Studies on Wiener filter design based on nonparametric models obtained from observed data also exist in the literature, see [Bh73], [Kar80], [BhKar83].

filter, and a performance degradation. An expression for the sensitivity of the estimator performance to variations in the optimal filter would be a useful tool for studying this effect. The question of sensitivity is also instrumental when the optimal filter is known but is too complex, and the use of a simplified filter structure may be preferred.

Few results on the sensitivity of the optimal Wiener design seem to be available in the literature. A deconvolution estimator partially inverts the system dynamics[2]. This property may possibly induce sensitivity to model errors. Intuitively, if the channel transfer function contains deep nulls, the optimal design will be sensitive due to the presence of peaks in the corresponding deconvolution filter gain.

A sensitivity analysis of realizable Wiener filters does not seem to be present in the literature. Studies on the effects of model errors exist for the unrealizable Wiener filter, see [VaPo83]. An obstacle to the analysis of realizable Wiener filters has been the causal bracket operation $\{\cdot\}_+$, required by the classical derivation of the solution, [AhSt94]. With the introduction of polynomial methods for linear quadratic design, pioneered by Kučera, [Ku79], further insights have been gained into the filtering mechanism. In [Gri85], some frequency domain properties of optimal filters were established. See, e.g. [Gri85], [AhSt91] and [AhSt94] for an account of the polynomial approach to filter design.

There do exist results on the sensitivity of Kalman–type filters (i.e. state–space structures and state estimation problems). Mainly, two types of model errors have been considered, namely noise covariance errors and uncertainties in the entries of the system configuration matrices. In 1968, Griffin and Sage ([GrSa68]) referred to the paper [Fag64] by Fagin as one of the first studies on the sensitivity of filtering techniques to (large) model errors. The analysis was conducted on the discrete–time Kalman filter. In [LePe73], the problem of quantifying the bias error due to parameter uncertainty in state estimates in the Kalman–Bucy filter was addressed. The authors report the Ph.D. dissertation of Pearson, [Pe71] as a survey on sensitivity studies. A more recent reference is [ToPa80], where bounds on the state estimation error given by a Kalman–type estimator are provided, based on the range of errors in the model matrices. Those bounds are obtained from complicated equations, and they will provide a measure of the performance degradation only after a filter design has been obtained. In principle, results obtained for the Kalman filter could be extended to the Wiener filter, which coincides with the Kalman filter in the stationary case. Unfortunately, a formulation of the problem in the state–space framework turns out to complicate the derivation of results useful in a transfer function framework.

2.2 Optimal Wiener Filter Design

In this section, we will summarize the optimal design of realizable Wiener filters based on polynomial methods as presented in [AhSt89]. Some additional insights and simplifications will also be presented.

[2] The presence of measurement noise modifies the full inversion, for instance, with the introduction of blocking zeros at the location of the stable poles of the noise spectrum.

The notation introduced in Section 1.3 is repeated here for the convenience of the reader. Data are assumed to be generated by the linear, time–invariant and stable system

$$\begin{aligned} y(t) &= G(q^{-1})u(t) + v(t) \\ v(t) &= H(q^{-1})e(t) \\ u(t) &= H_u(q^{-1})w(t) \end{aligned} \tag{2.2.1}$$

$$\lambda_w = \mathrm{E}w^2(t) \quad ; \quad \lambda_e = \mathrm{E}e^2(t) \quad ; \quad \lambda_u = \mathrm{E}u^2(t)$$

where the input signal $u(t)$ and the measurement noise $v(t)$ are stationary, real valued, zero–mean stochastic processes, represented by innovation models. The transfer functions $H_u(q^{-1})$ and $H(q^{-1})$ will thus be stable and minimum phase, with the first impulse response coefficient equal to one. The input and the noise are assumed to be mutually uncorrelated. For further use, introduce the following variance ratios:

$$\rho \triangleq \lambda_e/\lambda_w \quad ; \quad \rho_u \triangleq \lambda_u/\lambda_w \quad . \tag{2.2.2}$$

The stable transfer functions in (2.2.1) are expressed by polynomials in the backward shift operator q^{-1}:

$$\begin{aligned} G(q^{-1}) &= \frac{B(q^{-1})}{A(q^{-1})} = \sum_{k=0}^{\infty} g_k q^{-k} \\ H(q^{-1}) &= \frac{M(q^{-1})}{N(q^{-1})} = 1 + \sum_{k=1}^{\infty} h_k q^{-k} \\ H_u(q^{-1}) &= \frac{C(q^{-1})}{D(q^{-1})} \end{aligned}$$

where the polynomial $B(q^{-1})$, etc, are denoted as

$$B(q^{-1}) = b_0 + b_1 q^{-1} + b_2 q^{-2} + \ldots + b_{nb} q^{-nb} \quad .$$

The system (2.2.1) is represented by the quadruple

$$S = \{G, H, \rho, H_u\} \quad .$$

Consider the deconvolution scheme of Figure 2.1. The estimate of $u(t-m)$ is obtained from measurements of the output signal $y(t)$ up to time t, via a causal, stable and linear filter $F(q^{-1})$, as

$$\hat{u}(t-m) = F(q^{-1})y(t) \tag{2.2.3}$$

where m is the smoothing lag used in the estimation[3]. The estimation error $z(t)$ is then given by

$$z(t) = u(t-m) - F(q^{-1})y(t) \quad . \tag{2.2.4}$$

The objective of the design is to obtain the filter $F(q^{-1})$ which minimizes the mean–square error criterion

$$V(S,F) = \mathrm{E}z^2(t) \tag{2.2.5}$$

over the set of all causal, stable and linear filters[4].

[3]The case of linear prediction of $u(t)$, $m < 0$, is not considered.

[4]The fact that the filter $F(q^{-1})$ is applied on data generated by the system S is stressed by the notation $V(S,F)$.

The solution to the above problem is given by the realizable *Wiener Filter*, [AndMo79], [AhSt91], [AhSt94]. The Wiener filter for the system S will be indicated by

$$F_W(q^{-1})$$

and the corresponding performance by

$$V_W \triangleq V(S, F_W) \quad .$$

The Wiener filter $F_W(q^{-1})$ can be obtained from the following design equations, see [AhSt89]:

$$F_W(q^{-1}) = Q_1(q^{-1})\frac{A(q^{-1})N(q^{-1})}{\beta(q^{-1})} \tag{2.2.6}$$

$$r\beta\beta_* = BB_*CC_*NN_* + \rho MM_*AA_*DD_* \tag{2.2.7}$$

$$q^{-m}CC_*B_*N_* = r\beta_*Q_1 + qDL_* \tag{2.2.8}$$

where the polynomial spectral factor $\beta(q^{-1})$ in (2.2.7) is monic and stable, and the polynomials[5] $Q_1(q^{-1})$ and $L_*(q)$ are the unique solution of the Diophantine equation (2.2.8). For the existence of a stable estimator, the right–hand side of (2.2.7) must be positive on the unit circle.

Define the *normalized* innovation model of the output process $y(t)$ as

$$y(t) = \sqrt{\rho_y}H_y(q^{-1})w_y(t) \tag{2.2.9}$$

where $H_y(q^{-1})$ is monic and minimum phase, ρ_y is a scalar gain, used to normalize the innovation process $w_y(t)$ to have the same variance as $w(t)$ in (2.2.1), i.e.

$$\mathrm{E}w_y^2(t) = \lambda_w \quad .$$

Since $u(t)$ and $v(t)$ in (2.2.1) are mutually uncorrelated, the innovation model in (2.2.9) is, with the use of the spectral factorization (2.2.7), given by

$$H_y(q^{-1}) = \frac{\beta(q^{-1})}{A(q^{-1})N(q^{-1})D(q^{-1})}$$

$$\rho_y = r$$

where the scalar r is obtained from the spectral factorization (2.2.7). As in the classical formulation, the Wiener filter (2.2.6) can thus be partitioned by two filters, as

$$\boxed{F_W(q^{-1}) = F_{sh}(q^{-1})F_{wh}(q^{-1})} \tag{2.2.10}$$

The filter $F_{wh}(q^{-1})$ in (2.2.10) is the *whitening filter*

$$F_{wh}(q^{-1}) = \frac{1}{\sqrt{\rho_y}}\frac{1}{H_y(q^{-1})} = \frac{1}{\sqrt{r}}\frac{A(q^{-1})N(q^{-1})D(q^{-1})}{\beta(q^{-1})} \tag{2.2.11}$$

[5]Recall that $B_* = B_*(q)$ means conjugation of the polynomial $B(q^{-1})$. See Section 1.3.

which re–creates the normalized innovation process $w_y(t)$ in (2.2.9) from the measurement signal $y(t)$. Hence,

$$| F_{wh}(e^{j\omega}) |^2 = \frac{\lambda_w}{S_y(e^{j\omega})} \qquad (2.2.12)$$

where $S_y(e^{j\omega})$ is the output spectrum. The filter $F_{sh}(q^{-1})$ in (2.2.10) is the *shaping filter*

$$F_{sh}(q^{-1}) = \sqrt{r}\frac{Q_1(q^{-1})}{D(q^{-1})} \qquad (2.2.13)$$

which constructs an estimate of the input sequence $u(t-m)$ from the normalized innovation process $w_y(t)$. The shaping filter (2.2.13) will, as we shall see below, play an important role in the quantification of the attainable performance. Introduce the following parameter

$$\boxed{\alpha \triangleq \frac{1}{\pi}\int_0^{\pi} | F_{sh}(e^{j\omega}) |^2 \, d\omega} \qquad (2.2.14)$$

The parameter α represents the *average power gain*, or the H_2–norm, of the shaping filter. It determines the attainable performance of a Wiener filter, as stated by the following lemma.

Lemma 2.1 The performance of the Wiener estimator. The performance of the Wiener estimator (2.2.6) for the system S and for an arbitrary smoothing lag m is given by

$$V_W = \lambda_w(\rho_u - \alpha) \qquad (2.2.15)$$

where λ_w is the variance of the input innovation $w(t)$ in (2.2.1), ρ_u is defined in (2.2.2) and α is defined in (2.2.14).

Proof. See Appendix A, Section A.1 ■

Difficult deconvolution problems will result in shaping filters with low average gain α. In such a case, a Wiener estimator cannot improve much upon the performance of the trivial estimator $F(q^{-1}) \equiv 0$, for which

$$V(S,0) = \lambda_w \rho_u = \lambda_u \quad .$$

Expression (2.2.15) is a more compact and simpler alternative to the expressions for the optimal performance given, for instance, in [Gri85], [AhSt89] and [AhSt94].

Remark 2.1 Consider the case of pure filtering ($m = 0$) of a white input, i.e. let

$$H_u(q^{-1}) = \frac{C(q^{-1})}{D(q^{-1})} \equiv 1 \quad .$$

The parameter $\rho_u = \lambda_u/\lambda_w$ will thus be equal to one. The left hand side of (2.2.8) is then a polynomial in q with only positive powers and a constant term. The polynomial

$Q_1(q^{-1})$ can then not have terms with negative powers of q. Therefore, it will be a constant which is equal to

$$Q_1(q^{-1}) = \frac{b_0}{r}$$

where the scalar r is obtained from the spectral factorization (2.2.7). Since the denominator polynomial of $G(q^{-1})$ in (2.2.1) is monic, $b_0 = g_0$. The parameter α in (2.2.14) will therefore be given by

$$\alpha = \frac{g_0^2}{r} \tag{2.2.16}$$

and the performance of the Wiener filter $F_W(q^{-1})$ is

$$V_W = \lambda_w(1-\alpha) = \lambda_w\left(1 - \frac{g_0^2}{r}\right) \tag{2.2.17}$$

■

Consider the case of pure filtering ($m = 0$) of a white input, as in Remark 2.1. When noise $v(t)$ in (2.2.1) is not present, and the system is minimum phase, then the Wiener filter is the inverse of the channel $F_W(q^{-1}) = G^{-1}(q^{-1})$. It is well known that, when noise $v(t)$ in (2.2.1) is present, it will *decrease* the gain of the optimal filter. Hence, one could expect that, in general,

$$\mid F_W(e^{j\omega}) \mid \leq \mid G(e^{j\omega}) \mid^{-1} \quad . \tag{2.2.18}$$

The inequality in (2.2.18) can, in fact, be easily proved by means of Lemma 2.1.

Lemma 2.2 Let $F_W(q^{-1})$ be the Wiener filter designed for the system (2.2.1) with $m = 0$, and let

$$H_u(q^{-1}) = \frac{C(q^{-1})}{D(q^{-1})} \equiv 1$$

Then, the following inequality holds[6]:

$$\mid F_W(e^{j\omega}) \mid \leq \mid G(e^{j\omega}) \mid^{-1} \quad .$$

Proof. Follows from Lemma 2.1. See Appendix A, Section A.2 ■

It is worth noting that while the effect of the noise on the optimal filter design is very complicated, its effect on the optimal performance is simply described by the parameter α introduced in (2.2.14). For a given channel $G(q^{-1})$, several different noise spectra can thus be expected to result in the same optimal attainable performance.

[6]The same result can not be proved for an arbitrary smoothing lag or an arbitrary input spectrum. The problem is that in these cases the shaping filter is no longer a constant. See the proof.

2.3 The Sensitivity of the Wiener Filter

In this section, we will study the sensitivity of the optimal deconvolution filter performance with respect to unstructured perturbations of the optimal filter. The main result is given by Theorem 2.1 and Remark 2.2, where it is shown that the performance is determined by the relative (percentage) perturbation in the frequency domain of the optimal filter, via a simple integral relation. The criterion to consider in order to minimize the performance degradation of suboptimal filters is obtained as a result. Based on Theorem 2.1, the sensitivity of the optimal design can also be easily assessed. By means of computer simulations, it is shown in Example 2.1 that the performance of (high order) Wiener filters is, in general, not severely degraded when using well designed suboptimal filters of low order.

The notations are summarized in Section 2.2.

2.3.1 Sensitivity to Measurement Noise

First, we shall consider the effect that measurement noise $v(t)$ has on the attainable performance. For simplicity, let us investigate the case of pure filtering, ($m = 0$) and let the channel $G(q^{-1})$ be minimum phase[7]. Consider a (stable) filter $F(q^{-1})$ to be applied on data generated by the system S. The deconvolution filter can be rewritten as

$$F(q^{-1}) = \frac{\Delta(q^{-1})}{G(q^{-1})} \tag{2.3.1}$$

for some stable $\Delta(q^{-1})$. With the use of (2.2.3) and (2.2.4), the performance $V(S, F)$ can be expressed in the frequency domain, see Appendix A, Section A.1, as[8]

$$V(S,F) = \mathrm{E}\, z^2(t) = \frac{1}{\pi}\int_0^{\pi} \left\{ \mid 1 - FG \mid^2 S_u + \mid F \mid^2 S_v \right\} d\omega \tag{2.3.2}$$

where S_u and S_v are the input and the noise spectra, respectively. By inserting (2.3.1) into (2.3.2), the performance results as

$$V(S,F) = \frac{1}{\pi}\int_0^{\pi} \left\{ \mid 1 - \Delta \mid^2 S_u + \mid \Delta \mid^2 \frac{S_v}{\mid G \mid^2} \right\} d\omega \quad . \tag{2.3.3}$$

Expression (2.3.3) reveals the effect of noise on the attainable performance. The noise acts locally in the frequency domain to modify the full inversion of the channel transfer function $G(q^{-1})$. Indeed, when noise is not present, expression (2.3.3) is minimized by $\Delta \equiv 1$ in (2.3.1), and the optimal filter will just be the inverse of the channel. Introduce the following ratio

$$\mathrm{LCNR} \triangleq \frac{\mid G(e^{j\omega}) \mid^2}{S_v(e^{j\omega})} \tag{2.3.4}$$

where LCNR stands for the *local channel to noise ratio*. The lower the LCNR is in a particular frequency region, the more the factor Δ that minimizes (2.3.3) will differ

[7] The same results and conclusions obtained below apply to the case of arbitrary smoothing lags $m > 0$ and nonminimum phase channels.

[8] The argument $e^{j\omega}$ has been omitted for simplicity of notation.

from 1 in that region. In that case, the optimal attainable performance is expected to be degraded. The Wiener filter gain depends explicitly on the LCNR. This latter fact was illustrated in [Gri85]. The point that needs to be stressed is *the sensitivity of the filter performance to the* LCNR (2.3.4). A poor (optimal) filter performance may result even with a high signal to noise ratio, if the LCNR is low in some frequency regions. In particular, channels with deep nulls or the occurrence of noises with narrow spectral peaks are environments where liner deconvolution filters are not expected to be good candidates for the solution of deconvolution problems.

2.3.2 The Feasible Filter Set

Next, we shall determine the degradation in performance caused by using a filter $F(q^{-1})$, different from the Wiener filter $F_W(q^{-1})$, for the estimation of the input to the system S. Consider the following definition.

Definition 2.1 The Feasible Filter Set. For a given system S, the feasible filter set of ν–level, $\text{FFS}_\nu(S)$, is defined as the set of filters that result in a normalized[9] performance degradation not worse than ν.

$$\text{FFS}_\nu(S) \triangleq \left\{F(q^{-1}) : V(S,F) \leq V_W + \lambda_w \nu\right\} \tag{2.3.5}$$

■

For a polynomial $P(q^{-1})$, introduce the *polynomial spectral factor* $\bar{P}(q^{-1})$ as the *stable* polynomial obtained by the spectral factorization

$$\bar{P}(q^{-1})\bar{P}_*(q) = P(q^{-1})P_*(q) \quad . \tag{2.3.6}$$

Introduce the *minimum phase Wiener filter* $\bar{F}_W(q^{-1})$ as the filter given by

$$\bar{F}_W(q^{-1}) = \bar{F}_{sh}(q^{-1})F_{wh}(q^{-1}) \tag{2.3.7}$$

where $F_{wh}(q^{-1})$ is the whitening filter defined in (2.2.11), and $\bar{F}_{sh}(q^{-1})$ is the *minimum phase shaping filter* defined as

$$\bar{F}_{sh}(q^{-1}) \triangleq \sqrt{r}\frac{\bar{Q}_1(q^{-1})}{D(q^{-1})} \quad . \tag{2.3.8}$$

The polynomial $\bar{Q}_1(q^{-1})$ in (2.3.8) is the polynomial spectral factor of $Q_1(q^{-1})$, the solution of the Diophantine equation (2.2.8). From (2.2.13), observe that the following holds:

$$| \bar{F}_{sh}(e^{j\omega}) | = | F_{sh}(e^{j\omega}) | \quad . \tag{2.3.9}$$

[9]Since the optimal performance is $V_W = \lambda_w(\rho_u - \alpha)$, see Lemma 2.1, a corresponding normalization of the performance degradation via the power λ_w of the input innovation $w(t)$ is of more interest than the absolute value.

An arbitrary deconvolution estimator $F(q^{-1})$ can be expressed by a multiplicative perturbation $\Delta_F(q^{-1})$ of the Wiener filter $F_W(q^{-1})$ as

$$\boxed{F(q^{-1}) = F_W(q^{-1})\left[1 + \Delta_F(q^{-1})\right]} \tag{2.3.10}$$

with

$$\Delta_F(q^{-1}) = -F_W^{-1}(q^{-1})\left[F_W(q^{-1}) - F(q^{-1})\right] \quad . \tag{2.3.11}$$

The transfer function $\Delta_F(q^{-1})$ in (2.3.11) may be unstable, due to nonminimum phase zeros of $F_W(q^{-1})$. Introduce the stable perturbation $\tilde{\Delta}_F(q^{-1})$ given by

$$\tilde{\Delta}_F(q^{-1}) = -\bar{F}_W^{-1}(q^{-1})\left[F_W(q^{-1}) - F(q^{-1})\right] \tag{2.3.12}$$

where $\bar{F}_W(q^{-1})$ is the minimum phase Wiener filter defined in (2.3.7). Apparently, $\tilde{\Delta}_F(q^{-1})$ is obtained by reflecting the unstable poles of $\Delta_F(q^{-1})$ into the unit circle. With the use of (2.3.12), the deconvolution estimator $F(q^{-1})$ in (2.3.10) can then be expressed by the stable perturbation $\tilde{\Delta}_F(q^{-1})$ as

$$F(q^{-1}) = F_W(q^{-1}) + \bar{F}_W(q^{-1})\tilde{\Delta}_F(q^{-1}) \quad . \tag{2.3.13}$$

We can now state the following theorem, where the feasible filter set defined in (2.3.5) is specified in terms of the stable perturbation $\tilde{\Delta}_F(q^{-1})$.

Theorem 2.1 The MSE performance, when applying an arbitrary deconvolution estimator $F(q^{-1})$ on data generated by the system S, can be expressed as

$$V(S, F) = V_W + \lambda_w \delta_V \tag{2.3.14}$$

where δ_V represents a (normalized) degradation of the optimal performance. Then, the degradation δ_V is given by

$$\delta_V = \frac{1}{\pi}\int_0^{\pi} \mid \tilde{\Delta}_F \mid^2 \mid F_{sh} \mid^2 d\omega \tag{2.3.15}$$

where $\tilde{\Delta}_F(q^{-1})$ is the stable perturbation defined in (2.3.12) and where $F_{sh}(q^{-1})$ is the shaping filter defined in (2.2.13). The feasible filter set of ν–level for the system S is therefore given by the following set of filters:

$$\boxed{\text{FFS}_\nu(S) = \left\{F(q^{-1}) : \ \frac{1}{\pi}\int_0^{\pi} \mid \tilde{\Delta}_F \mid^2 \mid F_{sh} \mid^2 d\omega \leq \nu\right\}} \tag{2.3.16}$$

where ν is the prespecified performance degradation, as defined in (2.3.5).

Proof. See Appendix A, Section A.3 ■

Remark 2.2 The transfer function

$$\Delta_F(q^{-1})F_{sh}(q^{-1})$$

where $\Delta_F(q^{-1})$ is the multiplicative perturbation defined in (2.3.10), is stable, since the unstable poles of $\Delta_F(q^{-1})$ are canceled by the nonminimum phase zeros of $F_{sh}(q^{-1})$. Hence, the following integral is well defined

$$\frac{1}{\pi}\int_0^{\pi} \mid \Delta_F F_{sh} \mid^2 d\omega \quad .$$

With the use of the notation introduced above in (2.3.7)–(2.3.12), it is straightforward to see that the following relation holds

$$\frac{1}{\pi}\int_0^{\pi} \mid \tilde{\Delta}_F \mid^2 \mid F_{sh} \mid^2 d\omega = \frac{1}{\pi}\int_0^{\pi} \mid \Delta_F F_{sh} \mid^2 d\omega \quad . \tag{2.3.17}$$

Hence, the performance degradation in (2.3.15) is actually closely related to the degradation resulting from the multiplicative perturbation (2.3.10) ■

From Theorem 2.1 and Remark 2.2, an important conclusion can be drawn:

> The performance degradation that results from the use of a suboptimal filter is determined by the *relative perturbation of the optimal filter in the form of the unstructured multiplicative error* in (2.3.10), via the stable transfer function defined in (2.3.12), weighted by the shaping filter. Only the magnitude, not the phase, of the perturbation is of importance.

From (2.3.12), and with the use of (2.3.7), (2.3.9) and (2.2.12), the degradation of performance δ_V in (2.3.15) can be rewritten as

$$\boxed{\delta_V = \frac{1}{\lambda_w}\frac{1}{\pi}\int_0^{\pi} S_y \mid F_W - F \mid^2 d\omega} \tag{2.3.18}$$

The expression (2.3.18) represents the criterion to consider in order to minimize the performance degradation that results from the use of a suboptimal filter:

> Given a filter class, a suboptimal filter should be chosen to minimize the H_2–norm of the approximation error, weighted by the spectrum $S_y(e^{j\omega})$ of the measured signal $y(t)$.

The effect of the shaping filter in (2.3.15) and (2.3.16) is difficult to quantify exactly. However, a simple bound for the degradation of performance δ_V can be obtained by using the Cauchy–Schwarz inequality:

$$\delta_V = \frac{1}{\pi}\int_0^{\pi} \mid \tilde{\Delta}_F \mid^2 \mid F_{sh} \mid^2 d\omega \;\leq\; \left(\frac{1}{\pi}\int_0^{\pi} \mid \tilde{\Delta}_F \mid^2 d\omega\right)\cdot\left(\frac{1}{\pi}\int_0^{\pi} \mid F_{sh} \mid^2 d\omega\right) \quad .$$

With

$$V_\Delta \triangleq \frac{1}{\pi}\int_0^\pi \mid \tilde{\Delta}_F \mid^2 d\omega \tag{2.3.19}$$

the bound can be written as

$$\boxed{\delta_V \;\leq\; \alpha V_\Delta} \tag{2.3.20}$$

where α was defined in (2.2.14). Note that in the case of pure filtering ($m = 0$) of a white noise input, the inequality in (2.3.20) reduces to an equality, since the shaping filter is then constant. See Remark 2.1.

With the use of (2.3.20) and (2.3.16), a conservative version of the feasible filter set can be obtained as

$$\frac{1}{\pi}\int_0^\pi \mid \tilde{\Delta}_F \mid^2 d\omega \;\leq\; \frac{\nu}{\alpha} \quad . \tag{2.3.21}$$

The set of filters expressed as in (2.3.10), (2.3.13) that obey (2.3.21) is then contained in $\mathrm{FFS}_\nu(S)$.

From (2.3.20), another important conclusion can be drawn:

> Optimal filters that provide a good performance ($\alpha \approx \rho_u \geq 1$) are more sensitive to perturbations than optimal filters that provide a bad performance ($\alpha \approx 0$). In other words, the same relative perturbation will cause a larger performance degradation in the former case.

The conclusion above coincides with what could have been expected intuitively. With the use of Theorem 2.1 it can now be formally proved.

An interesting feature of the Wiener filter shown by the expression (2.3.15) is that its performance is equally sensitive to either an increase or a decrease of the filter magnitude. Apparently, the filters

$$\begin{aligned} F_1(q^{-1}) &= F_W(q^{-1})(1+\delta) \\ F_2(q^{-1}) &= F_W(q^{-1})(1-\delta) \end{aligned}$$

with a scalar $\delta > 0$, will result in the same performance degradation. That feature is characteristic of the optimal filter. It occurs for suboptimal filters only in particular cases. That is shown in Appendix A, Section A.4.

The expression (2.3.15) for the loss of performance caused by the use of a suboptimal filter allows for a quantification of the sensitivity of the optimal performance. That will be illustrated in the following subsection by means of computer simulations.

2.3.3 Quantification of the Sensitivity of Optimal Filters

In this subsection we will illustrate the sensitivity of optimal filters by means of computer simulations. Our interest will, in particular, be to study whether optimal filters of high order can be substituted by low order filters which can provide good deconvolution estimates. In the example below, (well designed) low–order approximations of the Wiener filter are obtained for a large number of different systems, and the resulting performance is investigated. The case of pure filtering ($m = 0$) of a white input is considered. The example will show that the optimal performance is, in general, only slightly degraded when using a well designed suboptimal filter of low order.

Example 2.1 For a given system S driven by a white input, for which the Wiener deconvolution estimator for filtering ($m = 0$) F_W is obtained, consider a filter class $\mathcal{F}$, such that

$$F_W \notin \mathcal{F} \quad .$$

Then, the following problem can, in principle, be solved:

$$\bar{V} = \min_{F \in \mathcal{F}} V(S, F) \quad . \tag{2.3.22}$$

Consider the expression (2.3.18) for the performance degradation caused by the use of a suboptimal filter for estimating the input value $u(t)$. With the use of the whitening filter (2.2.11), the expression can be rewritten as

$$\delta_V = \frac{1}{\pi} \int_0^{\pi} \left| \frac{1}{F_{wh}} \right|^2 \mid F_W - F \mid^2 d\omega \quad . \tag{2.3.23}$$

From (2.3.14) and (2.3.23), the filter $F_{opt}(q^{-1})$ that results in the performance $\bar{V}$ in (2.3.22) is obtained by solving

$$F_{opt} = \arg \min_{F \in \mathcal{F}} \frac{1}{\pi} \int_0^{\pi} \left| \frac{1}{F_{wh}} \right|^2 \mid F_W - F \mid^2 d\omega \quad . \tag{2.3.24}$$

An estimate $\hat{V}$ of $\bar{V}$ such as

$$\hat{V} \geq \bar{V}$$

can be obtained with the use of the following procedure, based on an identification algorithm, which provides a suboptimal solution of the minimization problem given in (2.3.24). The procedure is based on the estimation scheme depicted in Figure 2.2. The use of identification algorithms for the solution of approximation problems has been previously studied in the literature, see, for instance, [And*et*78], [Wah86].

Step 1 Generate a white sequence $\bar{u}(t)$ for $t = 1, \ldots, N_d$, and obtain the sequence $\bar{y}(t)$ as

$$\bar{y}(t) = F_W(q^{-1})\bar{u}(t) \quad .$$

Step 2 Use the whitening filter $F_{wh}(q^{-1})$ to generate the sequences

$$\begin{aligned} \bar{y}_f(t) &= F_{wh}^{-1}(q^{-1})\bar{y}(t) \\ \bar{u}_f(t) &= F_{wh}^{-1}(q^{-1})\bar{u}(t) \quad . \end{aligned}$$

Step 3 With an output error method, see [Lju87], [SödSt89], obtain the filter $\hat{F}(q^{-1})$ from the model equation

$$\bar{y}_f(t) = F(q^{-1})\bar{u}_f(t) + \bar{v}(t)$$

Figure 2.2: The scheme used in Example 2.1 for obtaining suboptimal filters when the optimal filter $F_W(q^{-1})$ is given. A suboptimal filter designed to minimize the variance of the residual error $\bar{v}(t)$ will minimize the corresponding loss of deconvolution performance.

by minimizing the loss–function

$$W_{id}(F) = \frac{1}{N_d} \sum_{1}^{N_d} \bar{v}^2(t) \quad . \tag{2.3.25}$$

Step 4 Asymptotically in the number of data, the loss–function (2.3.25) will coincide with the integral specified in the minimization problem (2.3.24). That easily follows from the equations in Steps 1–3 and the ergodic theorem, see ([Lju87], Chapter 9). The output error method may converge to local but nonglobal minima of the criterion (2.3.25). Hence, the quantity

$$\hat{V} \triangleq V(S, \hat{F}) \tag{2.3.26}$$

is then an estimate of $\bar{V}$ in (2.3.22) such as

$$\hat{V}_\Delta \ \geq \ \bar{V}_\Delta \quad .$$

The estimate $\hat{V}$ in (2.3.26) is given by the expression (2.3.2) computed for the filter $\hat{F}(q^{-1})$. Observe that the integral in (2.3.2) can be computed by solving a Diophantine equation for obtaining the positive real part of the integrand, see [Söd94].

Two sets of 50 systems, $\mathcal{S}_{ni}$ and $\mathcal{S}_{na}$, were randomly generated. The two sets will be further considered in the simulation studies of Chapter 5, where they will also be described in some more detail. See Section 5.4. In both sets, the channel transfer function $G(q^{-1})$ of each system S has an ARMA structure with polynomial degrees

$$nb = na = 12 \quad ,$$

and the noise is colored. The signal to noise ratio at the channel output is SNR = 13dB. For each system of both sets, the optimal filter $F_W(q^{-1})$ has an ARMA structure with equal polynomial degrees, denoted nF, such as

$$13 \ \leq \ nF \ \leq 22 \quad .$$

The systems have been selected so that the attainable performance when estimating a white input with unit variance is

$$V_W \ \leq \ 0.1 \quad .$$

Set $\mathcal{S}_{ni}$						
ν	order of suboptimal filters					
level	1	2	3	4	5	6
20 %	2	34	80	96	100	100
50 %	10	60	92	100	100	100
0.005	0	16	60	94	100	100
0.05	32	84	100	100	100	100

Set $\mathcal{S}_{na}$						
ν	order of suboptimal filters					
level	1	2	3	4	5	6
20 %	0	0	16	52	90	96
50 %	0	10	30	80	100	100
0.005	0	0	4	36	76	92
0.05	0	16	52	88	100	100

Table 2.1: Outcome of the experiment in Example 2.1. Left table: nice–systems $\mathcal{S}_{ni}$. Right table: nasty–systems $\mathcal{S}_{na}$. Percentage of filters resulting in a performance degradation less than ν, as compared to the optimal attainable performance. The ν–levels are indicated in the first column. First two rows: percentage degradation of performance. Last two rows: absolute degradation of performance. Each set is composed of 50 systems.

The two sets of systems $\mathcal{S}_{ni}$ and $\mathcal{S}_{na}$ differ in the way the zeros and the poles of the channel transfer functions were selected. Channels in the set $\mathcal{S}_{na}$, where the subscript "*na*" stands for "*nasty*", have poles and zeros closer to the unit circle than do the channels in the set $\mathcal{S}_{ni}$, where the subscript "*ni*" stands for "*nice*". The optimal filters for the systems in the set $\mathcal{S}_{na}$ are thus expected to be more difficult to be approximated by low order filters than those for the systems in the set $\mathcal{S}_{ni}$, due to the presence of peaks and notches in the transfer function. The former filters may then represent design cases more sensitive than the latter ones.

The procedure described above was utilized for obtaining low–order suboptimal filters and for estimating $\bar{V}$ in (2.3.22), with $N_d = 3000$. The suboptimal filters were chosen with an ARMA structure with equal polynomial degrees

$$n\hat{F} = 1, \ldots, 6 \quad .$$

For each system S_i, with $i = 1, \ldots, 50$, the procedure was repeated 3 times up to Step 3, and a suboptimal filter $\hat{F}_i(q^{-1})$ was obtained by averaging the parameters value of the filters at Step 3 over the repeated identification experiments. Then, the estimate $\hat{V}$ was computed as in Step 4 for the average filter $\hat{F}_i(q^{-1})$.

The experiment outcome is summarized in Table 2.1 for both sets. The value reported is the number of times, expressed in percentage, the suboptimal filters were inside the feasible filter set of ν–level for the system for which they are designed. In the first row, for instance, the value indicates the percentage of cases in which the performance of the suboptimal filters was inferior to the value $V_W(1+0.2)$, and in the third row, to the value $V_W + 0.005$. In Table 2.1, the designs corresponding to the set $\mathcal{S}_{na}$ show indeed to be more sensitive than those for the set $\mathcal{S}_{ni}$, as was expected. The use of a 4th order filter for the systems in $\mathcal{S}_{ni}$, and of a 5–6th order filter for those in $\mathcal{S}_{na}$, seems to guarantee only a slight degradation of the optimal performance.

The results of this experiment should be compared to those obtained in the simulation studies of Chapter 5. There, the systems in the two sets are assumed to be unknown to the designer, and the design is carried out from observed input–output data only ■

2.4 Approximate Modeling for Wiener Filter Design

In this section we consider the nominal design of Wiener filters without an exact model of the transmission channel and the noise correlation. Important is to determine the model precision required in order to maintain a sensible deconvolution performance, and whether the modeling stage can be optimized with respect to the total design.

Based on the results of the previous section, the effect of errors in the models will be related to perturbations of the optimal filter, as expressed in (2.3.11) and (2.3.12). With noisy data, the analysis is, unfortunately, very difficult, due to an obscure interaction of errors in the channel transfer function and in the noise spectrum. The problem of optimizing the modeling stage with respect to the global design has no easy solution, and a uniqueness problem will be pointed out. Important insights can be obtained in the noise–free case, where the effect of the model mismatch on the deconvolution performance is described by an integral measure in the frequency domain. Despite the difficulties in analysis, two methods for data–based (nominal) filter design will be proposed at the end of the section. The methods will be studied in Chapter 5 by means of computer simulations.

Data are assumed to be generated by the system (2.2.1), with input signal $u(t)$ assumed to be white. We will limit the analysis to the case of pure filtering ($m = 0$). The system (2.2.1) is represented by the triple

$$S_T = \{G_T, H_T, \rho_T\} \quad .$$

The exact description of S_T is not available, and a deconvolution filter for the system S_T is designed as the Wiener filter for an approximate model

$$S = \{G, H, \rho\} \quad .$$

The resulting filter will be denoted by

$$F_n \triangleq \mathcal{D}_n(S) \quad .$$

Other notations are summarized in Section 2.2. The subscript "T" is used to distinguish polynomials and other parameters related to the true system S_T from those related to the model S.

Consider the following definition.

Definition 2.2 The Feasible Model Set. For a given system S_T, the feasible model set of ν–level $\text{FMS}_\nu(S_T)$ is defined as the set of models for which the nominal design results in a normalized performance degradation not worse than ν

$$\text{FMS}_\nu(S_T) \triangleq \{S : V(S_T, F_n) \leq V_W + \lambda_w \nu\} \qquad (2.4.1)$$

■

See also the Definition 2.1 of the feasible filter set in the previous section. The feasible model set in the case of pure filtering ($m = 0$) of a white input is specified by the following lemma.

Lemma 2.3 For the design of a deconvolution estimator for filtering ($m = 0$) a white input with variance λ_w, the feasible model set of ν–level for a system S_T, as defined in (2.4.1), is given by the following set of models:

$$\text{FMS}_\nu(S_T) = \left\{ S : \alpha_T - 2\frac{g_0}{\hat{g}_0}\alpha + \alpha\frac{1}{\pi}\int_0^\pi \frac{| G_T |^2 \lambda_w + \rho_T | H_T |^2}{| G |^2 \lambda_w + \rho | H |^2} d\omega \leq \nu \right\} \tag{2.4.2}$$

where α_T and α, defined in (2.2.14), determine the performance of the Wiener filter for the system S_T

$$V_W = \lambda_w(1 - \alpha_T) \tag{2.4.3}$$

and for the model S

$$V(S, F_n) = \lambda_w(1 - \alpha)$$

respectively. The scalars g_0, for the system S_T, and $\hat{g}_0$, for the model S, are the first impulse response coefficients of the channel.

Proof. See Appendix A, Section A.5 ■

In the expression (2.4.2), the effect of modeling errors is mediated by both the parameter

$$\alpha = \hat{g}_0^2 / r \quad , \tag{2.4.4}$$

related via the scalar r to the spectral factorization (2.2.7), and the integral measure that involves the ratio of the true and the model output spectra. This coupled effect complicates further analysis, since modifications of the model will result in changes in both quantities. The combined effect is difficult to quantify.

A uniqueness problem should also be pointed out. In the considered case, the nominal filter is given by, see Remark 2.1,

$$F_n(q^{-1}) = \frac{\hat{g}_0}{r}\frac{A(q^{-1})N(q^{-1})}{\beta(q^{-1})} \quad . \tag{2.4.5}$$

For given polynomials $A(q^{-1})$ and $N(q^{-1})$, depending on the polynomial degrees na and nn, the spectral factorization (2.2.7) may have the same solution $\{r, \beta(q^{-1})\}$ for an infinite number of combinations of $B(q^{-1})$ and $M(q^{-1})$ and ρ, [Ah90]. Hence, the same Wiener filter will result from several different models. In loose terms, the models in the feasible model set do not need to be "close" to the system S_T. That property makes the feasible model hard to specify by quantifying a suitable range of model errors.

Remark 2.3 The effect of measurement noise is to lower the gain of the optimal filter as compared to a complete inversion of the channel. That effect was proved in Lemma 2.2

for the case of pure filtering ($m = 0$) of a white input. It is then reasonable to expect that for any model G of the channel G_T such as

$$| G(e^{j\omega}) | \leq | G_T(e^{j\omega}) | \quad , \quad \forall\, \omega \tag{2.4.6}$$

a noise spectrum can be found for which the nominal filter is "close" to the optimal one. The inverse of the model will in that case have a larger gain than the optimal filter. The noise spectrum can then be used as a "tuning knob", to be adjusted until a satisfactory design is obtained ■

If noise is not present and the channel model is minimum phase, then both α_T and α will be equal to one. The expression in (2.4.2) reduces to

$$1 - 2\frac{g_0}{\hat{g}_0} + \frac{1}{\pi}\int_0^{\pi} | G_T G^{-1} |^2 \, d\omega = \frac{1}{\pi}\int_0^{\pi} | 1 - G_T G^{-1} |^2 \, d\omega \leq \nu \tag{2.4.7}$$

where the relation (A.1.7) in Appendix A was used.

Corollary 2.1 Let the system be noise free and the true channel be minimum phase. Given a model class $\mathcal{G}$ where the true channel can not be described exactly, an optimal model is then, according to (2.4.7), obtained by solving the approximation problem

$$G_{opt} = \arg\min_{G \in \mathcal{G}} \frac{1}{\pi}\int_0^{\pi} | G_T |^2 | G_T^{-1} - G^{-1} |^2 \, d\omega \tag{2.4.8}$$

■

Next, we shall discuss strategies for approximate modeling of the channel and the noise correlation to serve for the design of nominal Wiener filters. We present two possible methods for data–based design of linear deconvolution filters. The first method is a mixed indirect–direct method, consisting of both channel modeling and direct filter adaptation. The second method is an indirect method based on a two–stage procedure.

A mixed indirect–direct method. The expression (2.4.2) shows that the loss of performance is explicitly determined by the estimated spectrum of the output signal $y(t)$,

$$\hat{S}_y(e^{j\omega}) = | G(e^{j\omega}) |^2 \lambda_w + \rho \, | H(e^{j\omega}) |^2 \quad .$$

Note that the integral in (2.4.2) resembles the criterion that is (asymptotically) minimized when fitting an ARMA model to the measured signal $y(t)$:

$$\begin{aligned} y(t) &= H_y(q^{-1})\epsilon(t) \\ \hat{H}_y &= \arg\min_{H_y \in \mathcal{H}_y} \mathrm{E}\epsilon^2(t) \end{aligned} \tag{2.4.9}$$

where $\mathcal{H}_y$ is a set of stable transfer functions with monic numerator and denominator, and

$$\mathrm{E}\epsilon^2(t) \triangleq W_\epsilon = \frac{1}{\pi}\int_0^{\pi} \frac{| G_T |^2 \lambda_w + \rho_T \, | H_T |^2}{| H_y |^2} \quad . \tag{2.4.10}$$

With the use of the ARMA model (2.4.9), the nominal filter (2.4.5) for a white input $u(t)$ can be expressed as

$$F_n(q^{-1}) = Q_1 H_y^{-1}(q^{-1}) \tag{2.4.11}$$

where

$$Q_1 = \frac{\hat{g}_0}{r} \tag{2.4.12}$$

is the nominal shaping filter. With the use of (2.3.14), (2.4.3), (2.4.4) and (2.4.12), the performance of the filter (2.4.11) will then be given by

$$\begin{aligned} V(S_T, F_n) &= (V_W + \lambda_w \delta_V) \\ &= \lambda_w(1-\alpha_T) + \lambda_w \left(\alpha_T - 2\frac{g_0}{\hat{g}_0}\frac{\hat{g}_0^2}{r} + \frac{\hat{g}_0^2}{r}\frac{1}{\pi}\int_0^{\pi} \frac{|\ G_T\ |^2 \lambda_w + \rho_T \mid H_T \mid^2}{r \mid H_y \mid^2} d\omega \right) \\ &= \lambda_w(1 - 2g_0 Q_1 + Q_1^2 W_\epsilon) \ . \end{aligned} \tag{2.4.13}$$

The expression (2.4.13) can be minimized with respect to W_ϵ and Q_1 independently. Moreover, the minimization is suitable to be based on observed data. In fact, by expressing the performance criterion (2.4.13) as

$$V(S_T, F_n) = \mathrm{E}\,[u(t) - Q_1 \epsilon(t)]^2$$

for a given residual sequence $\epsilon(t)$, obtained from an ARMA model (2.4.9), the optimal value of Q_1 is found as

$$Q_1 = \frac{\mathrm{E}u(t)\epsilon(t)}{W_\epsilon} \ . \tag{2.4.14}$$

The cross–covariance term in (2.4.14) can then be replaced by the sample estimate

$$\hat{\mathrm{E}}u(t)\epsilon(t) = \frac{1}{N_d}\sum_{t=1}^{N_d} u(t)\epsilon(t)$$

where N_d is the number of data available for the design. The design method that results from estimating an ARMA model for the output signal $y(t)$ and fitting the scalar gain Q_1 directly to the observed data will be named the *spectrum* method (spe–method), and denoted by $\mathcal{D}_s$. The method can still be applied with an arbitrary smoothing lag $m > 0$, where the shaping filter $Q_1(q^{-1})$ will be an FIR filter of order m. The deconvolution criterion is in this case no longer minimized by performing the minimization independently with respect to W_ϵ and to the shaping filter parameters. The reason is the presence in the criterion of cross–correlation terms between $u(t-m)$ and delayed residuals $\epsilon(t)$, the contributions of which are not minimized by minimizing W_ϵ.

The spe–method can be seen as solving the design equations of the Wiener filter directly based on the data. Estimation of the ARMA model corresponds to the spectral factorization, and results in the whitening filter. The estimation of the shaping filter corresponds to a modified Diophantine equation that takes the undermodeling into account.

In the case of pure filtering ($m = 0$) of a white input, the spe–method represents a clear alternative to the use of a *direct* method, when a filter is directly fit on the data in the model equation

$$u(t) = F(q^{-1})y(t) + z(t)$$

by minimizing the estimation error. Both methods optimize the deconvolution performance on the observed data sequence. With the spe–method, as with the direct method, a model for the channel is not estimated. That may constitute a drawback in certain cases, as discussed in Section 1.4.

The spe–method shows an interesting similarity with a method for suboptimal design of decision feedback equalizers that will be presented in Section 3.3. In that case, the indirect–direct method will consist of model based output predictions and direct adjustment of $m+1$ scalar gains ■

A two–stage method. In the previous section, the effect of noise on the attainable performance of a linear deconvolution filter was discussed. A good deconvolution performance can be attained when the parameter α_T is close to one. In those cases, the optimal filter is expected to be primarily determined by the channel inversion. Compare to Figures 5.1 and 5.2 in Section 5.4. If the noise level is reasonably estimated, the channel model will thus determine the design more than the noise model does. In the previous section, it was also shown that an optimal filter should be approximated in the H_2 norm weighted by the spectrum of the output signal $y(t)$, see (2.3.18). With the use of the whitening filter (2.2.11), the loss of performance can be rewritten as

$$\delta_V = \frac{1}{\pi}\int_0^{\pi} \left|\frac{1}{F_{wh}}\right|^2 \mid F_W - F \mid^2 d\omega \quad . \tag{2.4.15}$$

The bias error in the channel approximation can be controlled by filtering the data before the estimation, see ([Lju87], Chapter 13). The expression (2.4.15) suggests that it may be beneficial to use the inverse of the whitening filter as prefilter. In that way, the bias error will be reduced in frequency regions where a good approximation is required. The whitening filter is not available, but a *two–stage* design can be proposed. First a preliminary model is estimated. Then, the corresponding inverse whitening filter is utilized in the second stage to filter the data for the estimation of a new channel model. The nominal filter that results at the second stage is then utilized. This method will be named *filtered nominal* method, and will be denoted by $\mathcal{D}_f$ ■

The use of the two methods described above will be studied in Chapter 5 by means of computer simulations. The problem of estimating models and filters in direct and indirect methods will be further discussed in Chapter 4.

2.5 The Cautious Wiener Filter

In this section a robust method for the design of a linear deconvolution filter will be described and its use as the solution of the deconvolution problem defined in Section 1.3 discussed. The method was introduced by Sternad and Ahlén in [StAh93]. We will refer to it as the *cautious Wiener filter.* In Chapter 5, the method will be studied by means of computer simulations, and its utility will be compared to the nominal solutions discussed in the previous section.

Data are assumed to be generated by the system (2.2.1), with input signal $u(t)$ being white. The system $G(q^{-1})$ and the noise spectrum $S_v(e^{j\omega})$ are assumed to be only partially known; nominal models are provided together with an estimate of the size of the model uncertainty. A time invariant filter $F(q^{-1})$ is to be designed for estimating the input signal.

The information on the unknown system $G(q^{-1})$ consists of the knowledge of the system structure, given by

$$G(q^{-1}) = G_0(q^{-1}) + \Delta G(q^{-1}) = \frac{B_0(q^{-1})}{A_0(q^{-1})} + \frac{B_1(q^{-1})}{A_1(q^{-1})}\Delta B(q^{-1}) \quad (2.5.1)$$

$$\Delta B(q^{-1}) = \delta_{G0} + \delta_{G1}q^{-1} + \ldots + \delta_{Gn\delta}q^{-Gn\delta} \quad (2.5.2)$$

where the nominal model $G_0(q^{-1}) = B_0/A_0$ and B_1/A_1 in (2.5.1) are known stable transfer functions. The polynomial $\Delta B(q^{-1})$ represents a set of impulse response coefficients, described as zero mean stochastic variables with known second order moments:

$$\mathbf{P}_G = \bar{\mathrm{E}}\begin{bmatrix} \delta_{G0} \\ \vdots \\ \delta_{Gn\delta} \end{bmatrix}\begin{bmatrix} \delta_{G0} & \ldots & \delta_{Gn\delta} \end{bmatrix} > 0 \ . \quad (2.5.3)$$

Observe that the expectation operator over the distribution of the stochastic parameters in (2.5.2) has been denoted by $\bar{\mathrm{E}}[\cdot]$, to distinguish from the expectation $\mathrm{E}[\cdot]$ over the probability space induced by the noise process $v(t)$. The above information on $G(q^{-1})$ determines a confidence interval, specified pointwise in the frequency domain by

$$\mathcal{G}(e^{j\omega}) = \Big\{G(e^{j\omega}) = G_0(e^{j\omega}) + \Delta G(e^{j\omega}),\ \bar{\mathrm{E}}G(e^{j\omega}) = G_0(e^{j\omega}),$$
$$\bar{\mathrm{E}}[G(e^{j\omega}) - G_0(e^{j\omega})][G_*(e^{j\omega}) - G_{0_*}(e^{j\omega})] = \mathcal{P}_G(e^{j\omega})\Big\} \ . \quad (2.5.4)$$

Denote the sum of the diagonal elements in $\mathbf{P}_G$ by p_0, the sum of the elements in the ith super–diagonal by p_i, and that of the elements in the ith sub–diagonal by p_{-i}. Observe that $p_i = p_{-i}$. Then, the variance term in (2.5.4) results as

$$\mathcal{P}_G(q^{-1}, q) \triangleq \frac{B_{11}(q^{-1}, q)}{A_1(q^{-1})A_{1_*}(q)} \quad (2.5.5)$$
$$= \frac{B_1(q^{-1})B_{1_*}(q)}{A_1(q^{-1})A_{1_*}(q)}(p_{n\delta}q^{-n\delta} + \ldots + p_1q^{-1} + p_0 + p_1q + \ldots + p_{n\delta}q^{n\delta}) \ .$$

We will refer to the model of the unknown channel $G(q^{-1})$ given by (2.5.1)–(2.5.5) as a *second order model description* (SOMD). The SOMD is specified by the *nominal model* $G_0(q^{-1})$ and the *variance term* $\mathcal{P}_G(q^{-1}, q)$. Together, they define the confidence set (2.5.4). Discussions on the flexibility of the SOMD for describing several types of model uncertainties can be found in [StAh93], [Öhrn95].

Similarly as for $G(q^{-1})$, the a–priori information available on the noise spectrum $S_v(e^{j\omega})$ is given by the SOMD of the spectral factor $H(q^{-1})$:

$$\rho H(q^{-1}) = \rho_0 H_0(q^{-1}) + \Delta H(q^{-1}) = \rho_0\frac{M_0(q^{-1})}{N_0(q^{-1})} + \frac{M_1(q^{-1})}{N_1(q^{-1})}\Delta M(q^{-1}) \quad (2.5.6)$$

$$\Delta M(q^{-1}) = \delta_{H0} + \delta_{H1}q^{-1} + \ldots + \delta_{Hn\delta}q^{-Hn\delta} \quad (2.5.7)$$

$$\mathbf{P}_H = \bar{\mathrm{E}}\begin{bmatrix} \delta_{H0} \\ \vdots \\ \delta_{Hn\delta} \end{bmatrix}\begin{bmatrix} \delta_{H0} & \ldots & \delta_{Hn\delta} \end{bmatrix} \quad (2.5.8)$$

$$\mathcal{P}_H(q^{-1}, q) = \frac{M_{11}(q^{-1}, q)}{N_1(q^{-1})N_{1_*}(q)} \quad (2.5.9)$$

where the monic polynomials $M_0(q^{-1})$, $N_0(q^{-1})$ and $N_1(q^{-1})$ are stable[10]. The SOMD of $G(q^{-1})$ and $S_v(e^{j\omega})$ can be mutually correlated, with correlation between the coefficients in (2.5.2) and in (2.5.7). The design equation below, however, are not affected by such cross-correlation.

A reasonable measure of performance for a linear deconvolution estimator $F(q^{-1})$ is the variance of the estimation error

$$z(t) = u(t-m) - F(q^{-1})y(t) \tag{2.5.10}$$

averaged by the expectation operator $\bar{\mathrm{E}}[\cdot]$ over the set of admissible channel and noise models. The *cautious Wiener filter* is then defined as

$$F_C(q^{-1}) \triangleq \arg\min \bar{\mathrm{E}}\left[\mathrm{E}z^2(t)\right] \tag{2.5.11}$$

where the minimization in (2.5.11) is performed over the set of the linear, stable and causal filters. Define

$$\frac{\tilde{B}\tilde{B}_*}{\tilde{A}\tilde{A}_*} \triangleq \frac{B_0B_{0_*}}{A_0A_{0_*}} + \frac{B_{11}}{A_1A_{1_*}} \tag{2.5.12}$$

$$\tilde{\rho}\frac{\tilde{M}\tilde{M}_*}{\tilde{N}\tilde{N}_*} \triangleq \rho_0\frac{M_0M_{0_*}}{N_0N_{0_*}} + \frac{M_{11}}{N_1N_{1_*}} \tag{2.5.13}$$

where the polynomial $\tilde{A}(q^{-1})$, $\tilde{M}(q^{-1})$ and $\tilde{N}(q^{-1})$ are stable and monic, and $\tilde{\rho}$ is a nonnegative scalar. The cautious Wiener filter defined in (2.5.11) is then obtained, with the use of (2.5.12) and (2.5.13), from the following design equations, [StAh93]:

$$F_C(q^{-1}) = Q_1(q^{-1})\frac{\tilde{A}(q^{-1})\tilde{N}(q^{-1})}{\beta(q^{-1})} \tag{2.5.14}$$

$$r\beta\beta_* = \tilde{B}\tilde{B}_*\tilde{N}\tilde{N}_* + \tilde{\rho}\tilde{M}\tilde{M}_*\tilde{A}\tilde{A}_* \tag{2.5.15}$$

$$q^{-m}B_{0_*}A_{1_*}\tilde{N}_* = r\beta_*Q_1 + qL_* \tag{2.5.16}$$

where the polynomial spectral factor $\beta(q^{-1})$ in (2.5.15) is monic and stable, and the polynomials $Q_1(q^{-1})$ and $L_*(q)$ are the unique solution of the Diophantine equation (2.5.16). The design can be performed for more general problem descriptions as the one considered, for instance for colored inputs and multiple measurements, [ÖhAhSt95].

An important insight, discussed in [Öhrn95], on the design of the cautious Wiener filter is obtained by rearranging the terms in the spectral factorization (2.5.15) and the Diophantine equation (2.5.16). Define the *augmented noise model* as the canonical spectral factor obtained by the spectral factorization

$$\bar{\rho}\frac{\bar{M}\bar{M}_*}{(\tilde{N}A_1)(\tilde{N}_*A_{1_*})} \triangleq \rho_0\frac{M_0M_{0_*}}{N_0N_{0_*}} + \frac{M_{11}}{N_1N_{1_*}} + \frac{B_{11}}{A_1A_{1_*}} \quad . \tag{2.5.17}$$

Then, by inserting (2.5.17) into (2.5.15), the design equations can be rewritten as

$$r\beta\beta_* = B_0B_{0_*}(\tilde{N}A_1)(\tilde{N}_*A_{1_*}) + \bar{\rho}\bar{M}\bar{M}_*A_0A_{0_*} \tag{2.5.18}$$

$$q^{-m}B_{0_*}(\tilde{N}_*A_{1_*}) = r\beta_*Q_1 + qL_* \quad . \tag{2.5.19}$$

[10] All spectral factors in the set $H(q^{-1})$ in (2.5.6) are not canonical spectral factors of the noise process $v(t)$, due the possible presence of non–minimum phase terms when the coefficients of $\Delta M(q^{-1})$ vary in their probability space. In each case, an innovation model could however be obtained by a spectral factorization.

By comparing (2.5.14) and (2.5.17)–(2.5.19) to the Wiener filter design equations (2.2.5)–(2.2.7), the effect of the uncertainty described by the SOMD on the filter design is revealed. The cautious Wiener filter corresponds to the Wiener filter designed for the nominal channel model $G_0 = B_0/A_0$ when affected by noise with spectrum given by the augmented noise model (2.5.17). In (2.5.17), the contributions of the nominal noise model $H_0 = M_0/N_0$, the uncertainty in the system given by (2.5.5) and the uncertainty in the noise given by (2.5.9) are additive. As compared to the Wiener filter designed for the nominal model

$$S_0 = \{G_0, H_0, \rho_0\} \tag{2.5.20}$$

the cautious Wiener filter will thus have a lower gain, due to the augmented noise model in (2.5.17) having a spectral magnitude larger than the nominal one.

2.5.1 The Use in Data–Based Design

Next, we shall discuss upon the use of the cautious Wiener filter as a design method for obtaining a deconvolution filter from observed data. In Section 1.5, two basic and general problems, related to the use of robust design methods when models are obtained by system identification, were introduced. The first problem is that it is not clear how the SOMD can be inferred from data observations. Note that the nominal model (2.5.20) and transfer functions in (2.5.5) and (2.5.9) are required for the design. The second problem is that two classes of design problems, namely Case 1 and Case 2, have to be distinguished. The cautious Wiener filter solves a robust design problem for Case 1, while the design of filters from identified models corresponds to Case 2. The solution to a problem belonging to Case 1 is not guaranteed to be a good solution for a problem belonging to Case 2.

Recall that data are generated by the system (2.2.1) with a white input. Consider the use of a prediction error method, see [Lju87], [SödSt89], for estimating the model

$$\begin{aligned} y(t) &= \hat{G}(q^{-1})u(t) + \hat{H}(q^{-1})\hat{e}(t) && (2.5.21)\\ \mathrm{E}\hat{e}^2(t) &= \hat{\rho} && (2.5.22) \end{aligned}$$

based on a known sequence of inputs and outputs. If the polynomial orders of the transfer functions in (2.5.21) are precisely those of the polynomials in (2.2.1), then the model uncertainty will consist only of the variance of the models parameters. Under weak assumptions, the parameters of the estimated model will, asymptotically in the number of data, be distributed as gaussian stochastic variables

$$\hat{\theta} - \theta_T \approx \mathcal{N}\left(0, \frac{1}{N_d}\mathbf{P}\right) \tag{2.5.23}$$

where $\hat{\theta}$ denotes the parameter estimate, θ_T the parameters of the system (2.2.1) and N_d the number of data used for the identification experiment.

An estimate of the parameter variance can be obtained from data with the following estimator, ([SödSt89], p.207):

$$\hat{\mathbf{P}}_{N_d} = \frac{1}{N_d}\hat{\rho}\left[\frac{1}{N_d}\sum_{t=1}^{N_d}\psi(t,\hat{\theta})\psi^T(t,\hat{\theta})\right]^{-1} \tag{2.5.24}$$

where $\psi(t,\hat{\theta})$ denotes the gradient of the one–step ahead prediction of $y(t)$ based on the model (2.5.21) with respect to the parameters θ, calculated at the estimated value $\hat{\theta}$, and where $\hat{\rho}$ is the estimated variance (2.5.22) of the prediction error. The estimate given by (2.5.24) is an unbiased estimate of the (asymptotic) variance in (2.5.23). An algorithm for obtaining the nominal model (2.5.20) and the variance terms (2.5.5) and (2.5.9) of a SOMD, based on estimated models with given parameter variances, has been presented in [Öhrn95]. The algorithm is based on series expansions of the model transfer functions around the nominal value of the parameters.

In the undermodeled case, the model uncertainty consists of both variance and bias errors. Simple algorithms for uncertainty estimation are not available in the literature. See Section 1.5. The total model uncertainty can however be indicated by the variance of the parameters also in the undermodeled case. In [GuLju94], it is shown that for *validated* models obtained from an adequate number of data, the bias error is smaller than the variance error. The total model uncertainty can thus be assessed by increasing (e.g. doubling) the estimated variance of the parameters. One may still use the estimator (2.5.24) and then utilize the algorithm of [Öhrn95] to obtain the SOMD. Variance estimators for the undermodeled case have also been presented, see [HjaLju92], [Hja93].

Assume that a SOMD has been obtained from an estimated model and an estimate of the parameter variance. The cautious Wiener filter designed from the equations (2.5.14), (2.5.17)–(2.5.19) solves the (Case 1) design problem defined in (2.5.11) for the set of systems (2.5.4) and the corresponding set of noise models. It is not completely clear whether this approach is sensible also for problems belonging to Case 2. In [StAh93], cases are shown where the cautious design outperforms nominal designs also in the latter case. A reasonable design modification can be introduced by regarding the uncertainty descriptions (2.5.5) and (2.5.9) as "tuning knobs" to be adjusted until a satisfactory filter performance is obtained. A simple tuning is to utilize scalar gains γ_G and γ_H when computing the cautious noise model (2.5.17), as

$$\bar{\rho}\frac{\bar{M}\bar{M}_*}{(\tilde{N}A_1)(\tilde{N}_*A_{1_*})} \triangleq \rho_0\frac{M_0M_{0_*}}{N_0N_{0_*}} + \gamma_H\frac{M_{11}}{N_1N_{1_*}} + \gamma_G\frac{B_{11}}{A_1A_{1_*}} \quad . \tag{2.5.25}$$

The cautious filter thus results by selecting $\gamma_G = 1$ and $\gamma_H = 1$ in (2.5.25). However, the possible need of the tuning in (2.5.25) remains to be proven.

In summary, two points need to be investigated regarding the use of the cautious Wiener filter with identified models as described in this section:

- The possibility of obtaining an useful SOMD from the variance of the estimated parameters also in the case of undermodeling.
- The utility of a tuning via the scalar gains γ_G and γ_H in (2.5.25). It is worth noticing that if the tuning turns out to be advantageous, the use of the method may become complicated, due to the added difficulty of optimizing the deconvolution performance with respect to the scalar gains γ_G and γ_H.

We will address the above questions in Chapter 5, where the use of the cautious Wiener filter will be studied by means of computer simulations.

Chapter 3

The Decision Feedback Equalizer

3.1 Introduction

In this chapter, the use of the Decision Feedback Equalizer (DFE), [BePa79], [Meh90], [StAh90], as the solution of the deconvolution problem described in Section 1.3 will be analyzed. The DFE estimator is depicted in Figure 3.1.

In the next section, it is shown that the estimation mechanism of an optimal design can be seen as being based on two separated stages. In the first stage, optimal linear predictions of the output process $y(t)$ are calculated from past input–output values up to time $t - m - 1$, where m is the smoothing lag. Then, in the second stage, the estimate of $u(t - m)$ is obtained as the optimal linear mean square estimate based on the corresponding prediction errors. This result is important for two reasons. First, a new principle for the optimal design of DFEs is obtained, which leads to a novel method for suboptimal design. Second, the problem of designing approximate DFE schemes is clarified and the role played by a constraint on the filter structure can be explained. The above issues are analyzed in Section 3.3, where a filter structure to be used for suboptimal design of DFEs is also proposed. In the final section, strategies for approximate modeling to serve for DFE design are investigated. An optimal strategy is found in the case of filtering. The method can also be useful for the design of DFEs for smoothing, as will be illustrated by means of two examples in the final section.

3.1.1 An Overview

The linear equalizer based on a linear deconvolution filter was considered in the previous chapter. Its performance may be poor if the channel transfer function contains deep nulls. If the channel is nonminimum–phase, the performance of a linear equalizer can be bad even in the noise–free case. Much higher performance can be attained by nonlinear detectors. The best result is attained by maximum likelihood estimation of entire data sequences, usually implemented by the Viterbi algorithm, [For72]. However, it is well known that such detectors are complex, and that the complexity increases exponentially

Figure 3.1: The decision feedback equalizer, where m is the smoothing lag. Based on the received signal $y(t)$, an estimate $\bar{u}(t-m)$ of the transmitted digital symbol $u(t-m)$ is obtained. Previous decisioned symbols are utilized in the feedback path, to reduce the intersymbol interference. The resulting decision error is $\bar{z}(t) = u(t-m) - \bar{u}(t-m)$.

with the length of the system impulse response.

The algorithm complexity may be drastically reduced if a nonlinear algorithm is based on decisions made on a symbol–by–symbol basis. In that case, a good estimate of the transmitted symbol must be provided as the input to the decision block. The DFE is a very simple symbol–by–symbol detector. It consists of two linear filters, namely the feedforward filter $F_f(q^{-1})$ and feedback filter $F_b(q^{-1})$, and a nonlinear decision block where decisions are based on threshold levels. Previous symbol estimates are used to reduce the intersymbol interference which affects the received symbols. See Figure 3.1. DFE schemes have proven to be effective algorithms for the equalization of digital communication channels. They combine simplicity of implementation and operation with a performance comparable to that of maximum likelihood estimators, see, for instance, [Fa*etal*85].

The idea of using previous decisions to reduce the intersymbol interference was first introduced by Austin in 1967, [Au67]. In 1971, Monsen introduced two ideas fundamental for later studies on the DFE scheme, [Mon71]. First, the minimization of the mean–squares error (MSE) of the estimation *before* the nonlinear decision block was considered. Second, the analysis was conducted under the assumption of *correct past decisions*. In this way the nonlinear optimization that arises when considering minimization of the estimation error $\bar{z}(t)$, see Fig 3.1, is transformed into a linear quadratic optimization problem with respect to the filters structures. This simplified framework results in the scheme depicted in Figure 3.2. For such a scheme, Monsen obtained the design equations of the optimal DFE for a dispersive channel with the received signal affected by additive white measurement noise. The optimal solution did, however, not take the realizability constraint on the filters structure into account.

In our analysis we will utilize the framework introduced by Monsen. Hence, the *optimality* of the DFE is with respect to the *MSE criterion, before the decision block, under the*

Figure 3.2: Simplified scheme for the analysis of the decision feedback equalizer. Provided past decisions are correct, the scheme of Fig. 3.1 can be transformed to this equivalent structure. If the estimation error $z(t) = u(t-m) - \hat{u}(t-m)$ is considered instead of the decision error $\bar{z}(t)$, analysis and synthesis are simplified by removing the decision nonlinearity.

assumption of correct past decisions. The analysis is based on discrete–time baseband signals and models, sampled with the symbol rate. Of course, an optimal MSE design does not guarantee correct decisions after the nonlinear block. The possible occurrences of error bursts, due to the feedback structure, have received attention in the literature, see [CaBu76], [KeAnd87]. It is also worth mentioning that other design principles have been considered, for instance, based on the *zeros forcing condition*, [BePa79], and the *Minimum Probability of bit Errors* (MPE), [ShaYa74]. Monsen concluded that consideration of MPE and MSE lead to essentially the same probability of bit error.

The optimization of DFEs, using the mean–squares criterion, has been discussed repeatedly over the past twenty years, e.g. see [BePa79], [TroZe81], but the proposed design schemes either provided optimal filters that were not realizable or realizable filters with a suboptimal structure. Only in 1990, Sternad and Ahlén obtained the design equations of the optimal realizable DFE for general IIR channels and colored noises, see [StAh90]. The solution was obtained by assuming exact knowledge of the channel and noise models.

Exact modeling may not be assured in practice. That is true, in particular, in our basic scenario, where models are obtained by system identification, after observation of data sequences with known input. Furthermore, the optimal filters structure may be too complex, and the use of simplified structures may be preferred. Under such circumstances, it is important both to study strategies for suboptimal design and to understand the effect of a simplification of the filters on the estimation performance.

In Section 1.4, two basic approaches to filter design based on observed data sequences were introduced, namely *direct* and *indirect* methods. With a direct method, it is not clear what filters structure should be used, or how the choice of a certain structure influences the attainable MSE performance. With indirect methods, it is unclear how the models should be derived in order to optimize the total design, or whether meaningful solutions can be obtained by using a nominal design also in the undermodeled case. We will address these questions in the following sections.

3.2 The MSE Optimal DFE Revisited

In this section the MSE optimal DFE derived in [StAh90] will be reconsidered. The scheme of Figure 3.1 will be partitioned into a finer structure, which illustrates that the estimation can be seen as being based on two separated stages. This result leads to a new optimal design principle, given in Theorem 3.2. Its basic advantage, as compared to the algorithm of [StAh90], is that it can be easily modified to work when the exact models of the system and noise are not available, giving a superior performance in that case. The two–stage interpretation also clarifies the role that a simplification of the filters structure plays in the degradation of the MSE performance. The above aspects will be considered further in the next section.

The notation introduced in Section 1.3 is repeated here for the convenience of the reader. Data are assumed to be generated by the linear, time–invariant and stable system

$$\begin{aligned} y(t) &= G(q^{-1})u(t) + v(t) \\ v(t) &= H(q^{-1})e(t) \end{aligned} \tag{3.2.1}$$

$$\lambda_e = \mathrm{E}e^2(t) \quad ; \quad \lambda_u = \mathrm{E}u^2(t) = 1$$

$$\rho \triangleq \lambda_e / \lambda_u$$

where the noise process $v(t)$ is expressed in its innovation form. The input $u(t)$ is real valued, zero mean and white. Without loss of generality, it is assumed to have unit variance[1]. The input takes values on a finite alphabet $\mathcal{A}_u$, where the symbols are equally likely. The input is assumed to be uncorrelated with the noise. The transfer functions in (3.2.1) are expressed by polynomials in the backward shift operator q^{-1}:

$$\begin{aligned} G(q^{-1}) &= \frac{B(q^{-1})}{A(q^{-1})} = \sum_{k=0}^{\infty} g_k q^{-k} \\ H(q^{-1}) &= \frac{M(q^{-1})}{N(q^{-1})} = 1 + \sum_{k=1}^{\infty} h_k q^{-k} \end{aligned}$$

with polynomial degrees na, nb, etc. The system (3.2.1) is represented by the triple

$$S = \{G, H, \rho\} \quad .$$

Consider the DFE in Figure 3.1. Correct past decisions after the nonlinear block are assumed, see the previous section, so Figure 3.1 simplifies to Figure 3.2. The estimation error $z(t)$ is then given as

$$z(t) = u(t-m) - F_f(q^{-1})y(t) + F_b(q^{-1})u(t-m-1) \tag{3.2.2}$$

where m is the smoothing lag used in the estimation, and $F_f(q^{-1})$ and $F_b(q^{-1})$ are the feedforward and feedback filters, respectively. A given DFE design is indicated by the couple

$$F = \{F_f, F_b\} \quad .$$

[1] All the equations depend on the parameter ρ defined in (3.2.1). The mean–square error criterion (3.2.3) below can be normalized with the factor λ_u.

The objective of the design is to obtain the filters F that minimize the mean–square error criterion

$$V(S,F) = \mathrm{E}z^2(t) \tag{3.2.3}$$

over the set of all causal, stable and linear filters[2]. The solution of this problem, for general systems and noise structures, was first obtained in [StAh90]. The MSE optimal DFE is given by

$$\bar{F}_f(q^{-1}) = \frac{Q_f(q^{-1})}{R_f(q^{-1})} = \frac{\bar{S}_1(q^{-1})N(q^{-1})}{M(q^{-1})} \tag{3.2.4}$$

$$\bar{F}_b(q^{-1}) = \frac{Q_b(q^{-1})}{R_b(q^{-1})} = \frac{\bar{Q}(q^{-1})}{A(q^{-1})M(q^{-1})} \tag{3.2.5}$$

where $\bar{S}_1(q^{-1})$ and $\bar{Q}(q^{-1})$, together with polynomials[3] $\bar{L}_{1_*}(q)$ and $L_{2_*}(q)$, are the unique solution of the two coupled Diophantine equations

$$AM + q^{-1}\bar{Q} = q^m BN\bar{S}_1 + AM\bar{L}_{1_*} \tag{3.2.6}$$

$$qL_{2_*} = -\rho A_* M_* \bar{S}_1 + q^{-m} B_* N_* \bar{L}_{1_*} \tag{3.2.7}$$

with polynomial degrees

$$n\bar{s}_1 = n\bar{l}_1 = m \tag{3.2.8}$$

$$n\bar{q} = nl_2 = \max(na+nm, nb+nn) - 1 \ . \tag{3.2.9}$$

In the following, it will be shown that the design equations (3.2.4)–(3.2.9) are actually related to computing *optimal linear predictions* of the output process $y(t)$ from past input–output values up to time $t-m-1$. Then, the estimate of $u(t-m)$ is obtained as the *optimal linear mean square estimate* based on the prediction errors. First, the case of pure filtering ($m=0$) will be illustrated. The generalization of the results to an arbitrary smoothing lag $m>0$ is given in Theorem 3.2, where it is shown that the illustrated procedure provides the MSE optimal DFE, under the assumption of correct past decisions. We will make an extensive use of polynomial equations for obtaining optimal linear predictions of the output signal $y(t)$ from past input and output values, see, for instance, [Söd94]. These equations are summarized in Section B.1 of Appendix B.

With exact knowledge of the system (3.2.1), the output process $y(t)$ can be partitioned into a part that depends only on the signals at the present time instant t, and one that depends only on the past input and output values up to time instant $t-1$, [Söd94]. The partitioning is given by

$$y(t) = g_0 u(t) + e(t) + \hat{y}_o(t|t-1) \tag{3.2.10}$$

where $\hat{y}_o(t|t-1)$ is the optimal linear prediction, and

$$\tilde{y}_o(t|t-1) = g_0 u(t) + e(t) \tag{3.2.11}$$

is the corresponding prediction error. If the prediction error is available, it can be utilized for estimating the input $u(t)$, based on the very simple model (3.2.11). Observe that the optimal prediction error corresponds to the *least noisy measurement* of $u(t)$ that can be

[2]The fact that F is applied to data generated by the system S is stressed by the notation $V(S,F)$.
[3]Recall that $B_* = B_*(q)$ means conjugation of the polynomial $B(q^{-1})$. See Section 1.3.

available at time t. Because of the assumed correct past decisions, the past of $u(t)$ is available at time t. Thus, the optimal linear prediction can be computed as

$$\hat{y}_o(t|t-1) = F_{y,1}(q^{-1})y(t-1) + F_{u,1}(q^{-1})u(t-1) \ . \tag{3.2.12}$$

The filters $F_{y,1}(q^{-1})$ and $F_{u,1}(q^{-1})$ are obtained from the optimal predictor equations, see Section B.1, as

$$F_{y,1}(q^{-1}) \ = \ q\left[1 - H^{-1}(q^{-1})\right] \tag{3.2.13}$$

$$F_{u,1}(q^{-1}) \ = \ qH^{-1}(q^{-1})\left[G(q^{-1}) - g_0 H(q^{-1})\right] \ . \tag{3.2.14}$$

Hence, the value of $u(t)$ can be estimated from the optimal prediction error $\tilde{y}_o(t|t-1)$. From (3.2.11), it is seen that $u(t)$ is independent of past prediction errors. Then, the optimal linear mean–square estimator of $u(t)$ based on present and past prediction errors corresponds to the scalar estimator, see [Söd94],

$$\hat{u}(t) = \gamma \tilde{y}_o(t|t-1) \tag{3.2.15}$$

where γ is obtained by minimizing the MSE criterion (3.2.3)

$$\begin{aligned} V(S,\gamma) \ &= \ \mathrm{E}\left[u(t) - \gamma \tilde{y}_o(t|t-1)\right]^2 \\ &= \ 1 + \gamma^2(g_0^2 + \rho) - 2\gamma g_0 \ . \end{aligned} \tag{3.2.16}$$

The last equality follows from (3.2.11). The optimal gain and criterion values are found as

$$\bar{\gamma} \ = \ \frac{g_0}{g_0^2 + \rho} \tag{3.2.17}$$

$$V_{\mathrm{opt}} \ = \ \frac{\rho}{g_0^2 + \rho} \ . \tag{3.2.18}$$

Equations (3.2.11), (3.2.12) and (3.2.17) can now be arranged together, in order to obtain the feedforward and the feedback filters in (3.2.2). From (3.2.15), we obtain:

$$\hat{u}(t) = \bar{\gamma}\left[y(t) - F_{y,1}(q^{-1})y(t-1) - F_{u,1}(q^{-1})u(t-1)\right] \ . \tag{3.2.19}$$

Hence, the feedforward and the feedback filters of the DFE are obtained as

$$F_f(q^{-1}) \ = \ \frac{g_0}{g_0^2 + \rho}\left[1 - q^{-1}F_{y,1}(q^{-1})\right] \tag{3.2.20}$$

$$F_b(q^{-1}) \ = \ \frac{g_0}{g_0^2 + \rho}F_{u,1}(q^{-1}) \ . \tag{3.2.21}$$

The above filters may be expected to represent the optimal DFE for system (3.2.1), since they have been inferred from optimality considerations only. That is in fact true, as will be shown in Theorem 3.2 below.

The structure of the DFE scheme that results from the reasoning above is depicted in Figure 3.3. The estimate is obtained in two stages. In the first stage, the optimal prediction error is obtained. It represents the least noisy measurement of $u(t)$ available at time t, since the intersymbol interference is eliminated and the noise is reduced to the unavoidable innovation term $e(t)$. In the second stage, the best linear MSE estimate of $u(t)$, given the least noisy measurement $\tilde{y}_o(t|t-1)$, is obtained via the scalar gain $\bar{\gamma}$.

Figure 3.3: An alternative interpretation of the filtering mechanism in an optimal DFE. Pure filtering case ($m = 0$). The filters $F_{u,1}(q^{-1})$ and $F_{y,1}(q^{-1})$ serve for obtaining the optimal linear prediction of the received signal $y(t)$ as in (3.2.12). The scalar weight γ is adjusted by minimizing the MSE criterion (3.2.16).

Remark 3.1 The optimal DFE performance (3.2.18) can be compared to the performance of the Wiener filter (2.2.15). Note how the noise acts only via its innovation power (actually, the ratio between the noise innovation and the input power). The presence of noise correlation is fully utilized in the noise reduction. In fact, it can be easily shown that

$$\mathrm{E}v^2(t) \;\geq\; \mathrm{E}e^2(t)$$

where equality holds only in the case of white noise. For a fixed signal to noise ratio at the channel output, the worst case noise for the DFE is when the noise is white; then, correlation in the noise can not be utilized for noise reduction. The role of the system is also evident. The system dynamics is not inverted by the estimator. Instead, the noise model is to be inverted, see (3.2.4), (3.2.5) and (3.2.13), (3.2.14). The system frequency response has no influence in terms of performance. The intersymbol interference is assumed to be exactly canceled, and thus does not influence the MSE. What matters is only the first impulse response coefficient g_0. Hence, a clear guideline on when filtering can provide an adequate performance is easily provided. Corresponding insights could not be obtained from the analysis of the Wiener filter in the previous chapter ■

If a delay m is allowed when computing the input estimate at time t, several received samples influenced by $u(t-m)$ will be available. The estimation of $u(t-m)$ can then be based on the output values $y(t), \ldots, y(t-m)$, by repeating the same reasoning as above. The detailed structure of the DFE for one–step smoothing is depicted in Figure 3.4. As in the case of pure filtering, the input estimate is obtained in two stages. Multiple optimal linear predictions of the output signal are first calculated. The input estimate is then obtained as an optimally weighted sum of the corresponding prediction errors.

With an arbitrary smoothing lag m, $m+1$ optimal prediction errors

$$\tilde{y}_o(t-m-1+i|t-m-1) = y(t-m-1+i) - \hat{y}_o(t-m-1+i|t-m-1)$$

with $i = 1, \ldots, m+1$, are obtained. Then, the estimate of $u(t-m)$ is formed as the optimally weighted sum of the prediction errors, as

$$\hat{u}(t-m) = \bar{\gamma}_{m+1}\tilde{y}_o(t|t-m-1) + \ldots + \bar{\gamma}_1\tilde{y}_o(t-m|t-m-1) \qquad (3.2.22)$$

where the value of the vector of weights

$$\bar{\gamma} = \begin{bmatrix} \bar{\gamma}_{m+1} & \ldots & \bar{\gamma}_1 \end{bmatrix}^T$$

Figure 3.4: An alternative interpretation of the filtering mechanism in an optimal DFE. Smoothing lag $m = 1$. Compare to Fig 3.3.

is found by minimizing the MSE criterion (3.2.3). The optimal value of the vector of weights $\bar{\gamma}$ is given by the following theorem, where it is also shown that the estimate (3.2.22) is the optimal linear MSE estimate of $u(t-m)$ based on the optimal prediction errors that can be obtained at time t from the past of the input and output up to time $t-m-1$.

Theorem 3.1 Let the data be generated by the system (3.2.1). Consider the polynomial equations for obtaining the $(m+1)$–step ahead optimal linear prediction of $y(t)$, see Theorem B.1 in Appendix B,

$$B = P_{m+1}A + q^{-m-1}B_{m+1} \qquad (3.2.23)$$

$$M = T_{m+1}N + q^{-m-1}M_{m+1} \qquad (3.2.24)$$

with[4]

$$P_{m+1}(q^{-1}) = p_1 + p_2 q^{-1} + \ldots + p_{m+1} q^{-m} \qquad (3.2.25)$$

$$T_{m+1}(q^{-1}) = 1 + t_2 q^{-1} + \ldots + t_{m+1} q^{-m} \quad . \qquad (3.2.26)$$

Form the following matrices:

$$\mathbf{P} = \begin{bmatrix} p_1 & 0 & \ldots & 0 \\ p_2 & p_1 & \ddots & \\ \vdots & & \ddots & 0 \\ p_{m+1} & p_m & \ldots & p_1 \end{bmatrix} \qquad \mathbf{T} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ t_2 & 1 & \ddots & \\ \vdots & & \ddots & 0 \\ t_{m+1} & t_m & \ldots & 1 \end{bmatrix} \quad . \qquad (3.2.27)$$

Consider the following estimate of $u(t-m)$:

$$\hat{u}(t-m) = \bar{\gamma}_{m+1}\tilde{y}_o(t|t-m-1) + \ldots + \bar{\gamma}_1\tilde{y}_o(t-m|t-m-1) \qquad (3.2.28)$$

[4]The notations for the polynomials P_{m+1} and T_{m+1} differ from the standard notations, since the leading coefficients have subscript equal to 1 instead of 0. The reason is that the last coefficient has, in this way, subscript equal to $m+1$, where $m+1$ corresponds to the number of steps ahead in the prediction.

where $\tilde{y}_o(t-m-1+i|t-m-1)$ are the optimal i–step ahead linear prediction errors of $y(t)$ based on past input–output values up to time $t-m-1$. The vector of weights in (3.2.28)

$$\bar{\gamma} = \begin{bmatrix} \bar{\gamma}_{m+1} & \dots & \bar{\gamma}_1 \end{bmatrix}^T$$

is obtained as

$$\boxed{\bar{\gamma} = \left(\mathbf{P}^T\mathbf{P} + \rho\mathbf{T}^T\mathbf{T}\right)^{-1}\mathbf{P}^T\mathbf{r}} \tag{3.2.29}$$

with $\mathbf{r} = [0 \; \dots \; 0 \; 1]^T$.

Then, the estimate (3.2.28) is the optimal linear MSE estimate of $u(t-m)$, based on all of the optimal prediction errors

$$\tilde{y}_o(t-i|t-m-1-j) \;,\; i \;\geq 0 \;, j \;\geq 0 \;\;.$$

Proof. See Appendix B, Section B.2 ■

The main result of this section is given by the following theorem, where the reasoning illustrated above is summarized and is shown to result in an optimal DFE for the system (3.2.1).

Theorem 3.2 Let the data be generated by the system (3.2.1). Consider the following procedure. For Steps 1 and 2, refer to Theorem B.1 in Appendix B.

Step 1 For $i = 1, \dots, m+1$, solve, with respect to B_i, P_i, M_i, T_i, the polynomial equations of the optimal linear prediction of $y(t-m-1+i)$ based on the past input–output values up to time $t-m-1$,

$$B = P_iA + q^{-i}B_i \tag{3.2.30}$$
$$M = T_iN + q^{-i}M_i \tag{3.2.31}$$

with

$$nb_i = \max(nb, na) - 1$$
$$nm_i = \max(nm, nn) - 1$$

and

$$P_i(q^{-1}) = p_1 + p_2q^{-1} + \dots + p_iq^{-i} \tag{3.2.32}$$
$$T_i(q^{-1}) = 1 + t_2q^{-1} + \dots + t_iq^{-i} \;\;. \tag{3.2.33}$$

Step 2 For $i = 1, \dots, m+1$, compute the filters of the i–step optimal linear predictor

$$\hat{y}_o(t-m-1+i|t-m-1) = F_{y,i}(q^{-1})y(t-m-1) + F_{u,i}(q^{-1})u(t-m-1) \;\;.$$

The filters are given by

$$F_{y,i} = \frac{M_i}{M} \tag{3.2.34}$$
$$F_{u,i} = \frac{MB_i - M_iB}{MA} \;\;. \tag{3.2.35}$$

Step 3 Form the feedforward and feedback filters of the DFE (3.2.2) as

$$\begin{aligned} F_f(q^{-1}) &= \bar{\gamma}_{m+1} + \bar{\gamma}_m q^{-1} + \ldots + \bar{\gamma}_1 q^{-m} \\ &\quad - \left[\bar{\gamma}_1 F_{y,1}(q^{-1}) + \ldots + \bar{\gamma}_{m+1} F_{y,m+1}(q^{-1})\right] q^{-m-1} \quad (3.2.36) \\ F_b(q^{-1}) &= \bar{\gamma}_1 F_{u,1}(q^{-1}) + \ldots + \bar{\gamma}_{m+1} F_{u,m+1}(q^{-1}) \quad (3.2.37) \end{aligned}$$

with the vector of weights $\bar{\gamma}$ selected as specified by Theorem 3.1.

Then, the filters $F_f(q^{-1})$ and $F_b(q^{-1})$ so obtained minimize the MSE criterion (3.2.3) over the set of causal, stable and linear filters.

Proof. The proof follows by showing that the filters (3.2.36) and (3.2.37) coincide with the optimal filters obtained from the design equations (3.2.4)–(3.2.9). See Appendix B, Section B.3 ∎

On the basis of Theorem 3.2, a novel algorithm for the design of DFEs is obtained. It differs from (3.2.4)–(3.2.9) obtained in [StAh90], in the way the coefficients of the polynomials $\bar{S}_1(q^{-1})$ and $\bar{L}_{1*}(q)$ are calculated. In the proof of Theorem 3.2, it is shown that these coefficients can be calculated via the optimal value of the vector of weights $\bar{\gamma}$ as

$$\bar{\mathbf{s}} = \mathbf{T}\bar{\gamma} \quad (3.2.38)$$

$$\bar{\mathbf{l}} = -\mathbf{P}\bar{\gamma} + \mathbf{r} \quad (3.2.39)$$

where $\mathbf{r}$ is defined in Theorem 3.1, and

$$\bar{\mathbf{s}} = \begin{bmatrix} \bar{s}_0 & \ldots & \bar{s}_m \end{bmatrix}^T$$

$$\bar{\mathbf{l}} = \begin{bmatrix} \bar{l}_m & \ldots & \bar{l}_0 \end{bmatrix}^T$$

are the vectors formed with the coefficients of $\bar{S}_1(q^{-1})$ and $\bar{L}_{1*}(q)$, respectively. See Appendix B, Section B.3.

If the exact model description of the system (3.2.1) is available, equations (3.2.38) and (3.2.39) provide no advantage as compared to solving the polynomial equations (3.2.6) and (3.2.7). Their main importance is in cases where exact modeling cannot be assumed. That situation will be considered in the next section.

3.3 Suboptimal Design of DFEs

In this section we will consider the problem of designing DFEs when exact models of the transmission channel and the noise correlation are not available. Based on the results of the previous section, a suboptimal DFE design method is proposed for such a case. The method is first presented in Lemma 3.1 and Corollary 3.1. Next, it is shown that the method always attains equal or better performance than the use of a nominal design. Lemma 3.1 will also illustrate the effect of the use of a suboptimal DFE structure on the MSE performance. Guidelines for the design of DFEs in practice are proposed at the end of the section. The results of this section are of use for approximate modeling to serve for DFE design. The latter problem will be considered in the following section.

Data are assumed to be generated by the system (3.2.1), represented by the triple

$$S_T = \{G_T, H_T, \rho_T\} \quad .$$

The exact model description of S_T is not available, and the design of the DFE has to be carried out on the basis of an approximate model

$$S = \{G, H, \rho\} \quad .$$

The polynomials in the various transfer functions will be denoted with the standard notation of Section 1.3, see also Section 3.2. The subscript "T" is used to distinguish polynomials in the transfer functions of the system S_T from those in the model S.

The certainty equivalence principle is an obvious, and seemingly reasonable, design principle. The model is then utilized in the design equations (3.2.4)–(3.2.9), as if it were the correct description of the underlying system. In Section 1.2, this strategy was referred as the *nominal design* (nom–method), and will be denoted by

$$\mathcal{D}_n \quad .$$

An alternative design principle is suggested by the results of the previous section, where it was shown that the optimal estimate of $u(t-m)$ can be obtained in two stages. First, the optimal linear predictions of the output process $y(t)$ are computed, based on the past input–output values up to time $t-m-1$. Then, the estimate of $u(t-m)$ is obtained as the optimal linear mean square estimate based on the prediction errors. See Theorem 3.2. If the exact model of S_T is not available, the optimal predictions of $y(t)$ can not be obtained. Instead, suboptimal predictions are calculated from the model S. Denote them, for $i = 1, \dots, m+1$, by

$$\hat{\hat{\mathrm{y}}}(t-m-1+i|t-m-1) \tag{3.3.1}$$

and the corresponding prediction errors by

$$\tilde{\hat{\mathrm{y}}}(t-m-1+i|t-m-1) = y(t-m-1+i) - \hat{\hat{\mathrm{y}}}(t-m-1+i|t-m-1) \quad . \tag{3.3.2}$$

Then, the estimate of $u(t-m)$ can still be formed with the use of the suboptimal prediction errors in (3.3.2), as

$$\hat{\mathrm{u}}(t-m) = \gamma_{m+1}\tilde{\hat{\mathrm{y}}}(t|t-m-1) + \ldots + \gamma_1\tilde{\hat{\mathrm{y}}}(t-m|t-m-1) \tag{3.3.3}$$

where the vector of weights γ is obtained by minimizing the MSE criterion

$$V(\gamma) = \mathrm{E}[u(t-m) - \hat{\mathrm{u}}(t-m)]^2 \quad . \tag{3.3.4}$$

An open question is how the predictions in (3.3.1) should be chosen. This will, of course, depend on the information available for the design. An interesting interpretation of the resulting estimate can be obtained if the predictions are chosen as being the optimal linear predictions based on the model S. The certainty equivalence principle is thus applied when computing the predictions. In that case, the three–step procedure described in Theorem 3.2 can be repeated with a slight change in step 3. In steps 1 and 2, the model is utilized as if it were the correct description of the underlying system. In step 3, the value of the vector of weights $\bar{\gamma}$ which minimizes the corresponding MSE criterion is given in Lemma 3.1 below. This design method will be named the *multiple prediction method* (mp–method), and will be denoted by

$$\mathcal{D}_p \quad .$$

The DFE filters obtained with this method are given in Corollary 3.1 below.

Introduce the following notation. The optimal linear predictions based on the model S will be denoted, for $i = 1, \ldots, m+1$, by

$$\hat{y}(t-m-1+i|t-m-1) \tag{3.3.5}$$

and the corresponding prediction errors by

$$\tilde{y}(t-m-1+i|t-m-1) = y(t-m-1+i) - \hat{y}(t-m-1+i|t-m-1) \quad . \tag{3.3.6}$$

The estimate of $u(t-m)$ is obtained as a linear combination of prediction errors

$$\hat{u}(t-m) = \gamma_{m+1}\tilde{y}(t|t-m-1) + \ldots + \gamma_1\tilde{y}(t-m|t-m-1) \quad . \tag{3.3.7}$$

Lemma 3.1 Let the matrices **P** and **T** defined in Theorem 3.1 be obtained on the basis of the model S as if the model were the correct description of the system S_T. Define the model residual $\hat{e}(t)$ as

$$\hat{e}(t) = H^{-1}(q^{-1})\left[y(t) - G(q^{-1})u(t)\right] \tag{3.3.8}$$

where the transfer function $H(q^{-1})$, which describes the noise correlation, is assumed to be minimum phase. Introduce the following correlation matrices:

$$\mathbf{R}_{\hat{e}} = \mathrm{E}\ \hat{\mathbf{e}}\hat{\mathbf{e}}^T \tag{3.3.9}$$
$$\mathbf{R}_{\hat{e}u} = \mathrm{E}\ \hat{\mathbf{e}}\mathbf{u}^T \tag{3.3.10}$$

where

$$\begin{aligned} \mathbf{u} &= [\ u(t) \ \ \ldots \ \ u(t-m)\ ]^T \\ \hat{\mathbf{e}} &= [\ \hat{e}(t) \ \ \ldots \ \ \hat{e}(t-m)\ ]^T \quad . \end{aligned}$$

Consider the estimate $\hat{u}(t-m)$ of $u(t-m)$ given in (3.3.7). Then, for the considered estimator structure, the vector of weights

$$\gamma = [\ \gamma_{m+1} \ \ \ldots \ \ \gamma_1\ ]^T$$

in (3.3.7) for which the MSE criterion

$$V(\gamma) = \mathrm{E}[u(t-m) - \hat{u}(t-m)]^2 \tag{3.3.11}$$

is minimized is given by

$$\boxed{\bar{\gamma} = \left[\mathbf{P}^T\mathbf{P} + \mathbf{T}^T\mathbf{R}_{\hat{e}}\mathbf{T} + \mathbf{T}^T\mathbf{R}_{\hat{e}u}\mathbf{P} + \mathbf{P}^T\mathbf{R}_{\hat{e}u}^T\mathbf{T}\right]^{-1}\left[\mathbf{P}^T + \mathbf{T}^T\mathbf{R}_{\hat{e}u}\right]\mathbf{r}} \tag{3.3.12}$$

with $\mathbf{r} = [0 \ \ldots \ 0 \ 1]^T$.

Proof. See Appendix B, Section B.4 ■

Remark 3.2 If the model S is the exact description of the system S_T, the model residual (3.3.8) will coincide with the innovation process $e(t)$ in (3.2.1). Then the result of the lemma reduces to that of Theorem 3.1. Hence, the degradation of the optimal performance is *completely determined* by the use of suboptimal predictions when forming the input estimate (3.3.7). In the proof of Lemma 3.1, it is shown that this fact holds when any suboptimal predictions (3.3.1) are utilized. Prediction of the received signal $y(t)$ up to $m+1$ steps ahead is thus instrumental in the DFE mechanism. We will further comment on the implications of Lemma 3.1 on the design of suboptimal DFEs at the end of this section ■

Remark 3.3 The mp–method is suitable for data–based design. The correlation matrices $\mathbf{R}_{\hat{e}}$ and $\mathbf{R}_{\hat{e}u}$, that serve for the computation of the vector of weights $\bar{\gamma}$ in (3.3.12), can then be replaced by the sample estimates

$$\hat{\mathbf{R}}_{\hat{e}} = \frac{1}{N_d - m} \sum_{t=m+1}^{N_d} \hat{\mathbf{e}}_t \hat{\mathbf{e}}_t^T \tag{3.3.13}$$

$$\hat{\mathbf{R}}_{\hat{e}u} = \frac{1}{N_d - m} \sum_{t=m+1}^{N_d} \hat{\mathbf{e}}_t \mathbf{u}_t^T \tag{3.3.14}$$

where the subscript "t" as been used with obvious meaning in the vectors $\hat{\mathbf{e}}$, $\mathbf{u}$ defined in Lemma 3.1. The multiple prediction method will be studied experimentally in Chapter 5 by means of computer simulations ■

In the next Corollary, the multiple prediction method is summarized.

Corollary 3.1 The multiple prediction method. The DFE filters that correspond to the input estimate (3.3.7), with the vector of weights $\bar{\gamma}$ given in Lemma 3.1, have the following structure

$$\boxed{\begin{aligned} F_f(q^{-1}) &= \frac{N(q^{-1})}{M(q^{-1})} S_1(q^{-1}) \\ F_b(q^{-1}) &= \frac{Q(q^{-1})}{M(q^{-1})A(q^{-1})} \end{aligned}} \tag{3.3.15}$$

where the polynomials $S_1(q^{-1})$ and $Q(q^{-1})$, together with a polynomial $L_{1_*}(q)$, are defined as

$$\begin{aligned} S_1(q^{-1}) &= s_0 + s_1 q^{-1} + \ldots + s_m q^{-m} \\ L_{1_*}(q) &= l_0 + l_1 q + \ldots + l_m q^m \\ Q(q^{-1}) &= q_0 + q_1 q^{-1} + \ldots + q_{nq} q^{-nq} \end{aligned}$$

with

$$nq = \max(nb + nn, na + nm) - 1 \quad . \tag{3.3.16}$$

The polynomials $Q(q^{-1})$, $S_1(q^{-1})$ and $L_{1_*}(q)$ satisfy the following polynomial equation

$$q^{-1}Q = NBq^m S_1 + MA(L_{1_*} - 1) \quad . \tag{3.3.17}$$

The coefficients of the polynomials S_1 and L_{1_*} are given by

$$\boxed{\begin{aligned} \mathbf{s} &= \mathbf{T}\bar{\gamma} \\ \mathbf{l} &= -\mathbf{P}\bar{\gamma} + \mathbf{r} \end{aligned}} \tag{3.3.18}$$

where

$$\begin{aligned} \mathbf{s} &= [\ s_0 \quad \dots \quad s_m\ ]^T \\ \mathbf{l} &= [\ l_m \quad \dots \quad l_0\ ]^T \\ \mathbf{r} &= [\ 0 \quad \dots \quad 0 \quad 1\ ]^T \end{aligned}$$

and where the weight vector $\bar{\gamma}$ is given by Lemma 3.1.

Proof. Follows directly from the proof of Lemma 3.1 ■

Remark 3.4 The filters obtained with the multiple prediction method and with the nominal design have the same structure. Compare (3.3.15)–(3.3.17) to (3.2.4)–(3.2.6). The two designs differ in the value assigned to the coefficients of the polynomials S_1 and L_{1_*}. Observe that in both cases the coefficients are calculated via the relations

$$\mathbf{s} = \mathbf{T}\gamma \tag{3.3.19}$$

$$\mathbf{l} = -\mathbf{P}\gamma + \mathbf{r} \quad . \tag{3.3.20}$$

With the nom–method, the value of γ given by Theorem 3.1 as if the model S were the correct description of the system S_T is used. With the mp–method, the value $\bar{\gamma}$ in (3.3.12) is chosen, such that the MSE criterion (3.3.11) is minimized. Hence, the latter method will always provide equal or better performance than the nom–method. This result is formally stated in the next Lemma ■

Lemma 3.2 Let $V(S_T, F)$ be the performance of the DFE F when applied to data generated by the system S_T. Let F_n and F_p be the DFE designed on the basis of the model S with the nom–method and the mp–method, respectively. Then,

$$V(S_T, F_p) \leq V(S_T, F_n) \quad .$$

Proof. Follows from Theorem 3.2 and Lemma 3.1. See Remark 3.4 ■

3.3.1 On the Structure of Suboptimal DFEs

Assume that suboptimal predictions (3.3.1) or (3.3.5) of the received signal $y(t)$ are obtained from a certain predictor design. Then, an important question is whether estimators of $u(t-m)$ with a more complex structure than those given in (3.3.3) and (3.3.7) should be considered. The most general structure for a linear estimator based on the output prediction errors is given by

$$\hat{\mathrm{u}}(t-m) = \Gamma_{m+1}(q^{-1})\hat{\tilde{\mathrm{y}}}(t|t-m-1) + \ldots + \Gamma_1(q^{-1})\hat{\tilde{\mathrm{y}}}(t-m|t-m-1) \qquad (3.3.21)$$

where each $\Gamma_i(q^{-1})$ is a linear and stable filter. The use of scalar weights γ:s is motivated only when optimal linear predictions of $y(t)$ are available. It may hence seem the case that the estimator (3.3.21) should then be used in the DFE when only suboptimal output predictions can be obtained.

Lemma 3.1 suggests that the estimator (3.3.21) should *not* be considered. A suboptimal DFE should be designed in order to provide an input estimate in the previously discussed form

$$\hat{\mathrm{u}}(t-m) = \gamma_{m+1}\hat{\tilde{\mathrm{y}}}(t|t-m-1) + \ldots + \gamma_1\hat{\tilde{\mathrm{y}}}(t-m|t-m-1) \qquad (3.3.22)$$

where the scalar weights γ:s are adjusted by minimizing the MSE criterion (3.3.4). The reason is that Lemma 3.1 shows that the degradation of the optimal performance, when designing the DFE with the mp–method, is *completely determined* by the use in (3.3.22) of the suboptimal predictions (3.3.5). The same holds when arbitrary predictions (3.3.1) are used. See Remark 3.2.

For a given choice of (suboptimal) predictors

$$\hat{\tilde{\mathrm{y}}}(t-m+1+i|t-m-1) = \hat{F}_{y,i}(q^{-1})y(t-m-1) + \hat{F}_{u,i}(q^{-1})u(t-m-1) \qquad (3.3.23)$$

with $i = 1, \ldots, m+1$, the estimator structure (3.3.21) will result in the following DFE filters:

$$\begin{aligned} F_f(q^{-1}) &= \Gamma_{m+1}(q^{-1}) + \Gamma_m(q^{-1})q^{-1} + \ldots + \Gamma_1(q^{-1})q^{-m} \\ &\quad - \left[\Gamma_1(q^{-1})F_{y,1}(q^{-1}) + \ldots + \Gamma_{m+1}(q^{-1})F_{y,m+1}(q^{-1})\right] q^{-m-1} \qquad (3.3.24) \\ F_b(q^{-1}) &= \Gamma_1(q^{-1})F_{u,1}(q^{-1}) + \ldots + \Gamma_{m+1}(q^{-1})F_{u,m+1}(q^{-1}) \ . \qquad (3.3.25) \end{aligned}$$

Compare to Theorem 3.2. With the scalar estimator (3.3.22), the filters $\Gamma_i(q^{-1})$ are substituted by the scalar weights γ:s. Apparently, the DFE filters determined by the choice of the scalar estimator (3.3.22) will have a lower order then those in (3.3.24)–(3.3.25). With the use of the scalar estimator (3.3.22), the design of the predictor filters in (3.3.23) could then be reformulated with filters of higher order. This will, in general, improve the predictions of $y(t)$. The latter design is thus expected to attain a better performance than the one based on the complex estimator (3.3.21) because of Lemma 3.1.

In summary, modifications of the mp–method should be considered when designing suboptimal DFEs. The modifications are on the choice of the suboptimal predictions to be used when forming the estimate (3.3.22). They do not necessarily have to be designed as being optimal for the model S. The choice depends on the information available for the design, and on the structure constraints on the DFE filters. The structure of the

Figure 3.5: The structure for the design of decision feedback equalizers for one–step smoothing. The filters $F_{u,i}(q^{-1})$ and $F_{y,i}(q^{-1})$, $i = 1, 2$, serve for obtaining predictions of the received signal $y(t)$ as in (3.3.23). The estimate $\hat{u}(t-1)$ of $u(t-1)$ is formed as in (3.3.2), (3.3.3). The scalar weights γ:s are adjusted by minimizing the MSE criterion (3.3.4).

suboptimal DFE that result from the design guidelines proposed above is illustrated in Figure 3.5 for the case of one–step smoothing. The figure has been obtained from Figure 3.4 by rearranging the various blocks. The DFE structure to be used for an arbitrary smoothing lag $m > 1$ can be inferred from Figure 3.5 by simply adding the corresponding predictors filters and scalar weights γ:s.

3.4 Approximate Modeling for DFE Design

Strategies for obtaining approximate models to serve for DFE design will be studied in this section for the nominal design and for the multiple prediction method, introduced in the previous section. It will be shown that in the case of pure filtering ($m = 0$), the modeling stage can be optimized with respect to the total design. In general, this result does not hold for arbitrary smoothing lags, and we show why. The suboptimal design methods will be illustrated by means of two examples in the next section.

We recall that the analysis is carried out with respect to the simplified DFE scheme depicted in Fig 3.2, which results from the assumption of correct past decisions after the nonlinear block.

The problem of obtaining approximate models to serve for DFE design is stated as follows.

Problem 3.1 Approximate modeling for DFE design. Data are generated by the system $S_T = \{G_T, H_T, \rho_T\}$ in (3.2.1) and a DFE for the system S_T has to be designed. A DFE design is denoted by $F = \{F_f, F_b\}$. The system S_T is described by a model $S = \{G, H, \rho\}$ chosen from a model class

$$\mathcal{M} = \{\mathcal{G}, \mathcal{H}\}$$

for which at least one of

$$\begin{aligned} G_T(q^{-1}) &\notin \mathcal{G} \\ H_T(q^{-1}) &\notin \mathcal{H} \end{aligned}$$

holds. For a model S, the filters F are designed with either the nominal design or the multiple prediction method. See the previous section. This is denoted by

$$\begin{aligned} F_n &= \mathcal{D}_n(S) \\ F_p &= \mathcal{D}_p(S) \quad . \end{aligned}$$

The approximate modeling problem can then be formulated as obtaining

$$\boxed{\hat{S} = \arg\min_{S \in \mathcal{M}} V(S_T, \mathcal{D}(S))} \qquad (3.4.1)$$

for the relevant design rule $\mathcal{D}(S)$ ■

Remark 3.5 The polynomial orders of the filters F_f and F_b has been fixed by the choice of the model class, via the expression (3.3.15). The set

$$\mathcal{F} = [\mathcal{F}_f \times \mathcal{F}_b]$$

is then defined to denote the set of *all* the filters with the corresponding orders. Notice that the set of filters that can be obtained by the design methods above form only a *proper subset* of $\mathcal{F}$, since only filters with a certain structure are considered. In principle, the following problem:

$$\boxed{F_{opt} = \arg\min_{F \in \mathcal{F}} V(S_T, F)} \qquad (3.4.2)$$

can be solved with a *direct method.* See Section 1.4. It is not guaranteed that

$$V(S_T, \mathcal{D}(\hat{S})) = V(S_T, F_{opt}) \qquad (3.4.3)$$

■

In the following theorem, we show that Problem 3.1 can be solved for the multiple prediction method (mp–method) in the filtering case ($m = 0$). The use of the nominal design (nom–method) leads, instead, to an intractable optimization problem. Hence, there are two different contributions to the loss in filtering performance when using the nom–method. The first is given by Lemma 3.2: its performance is never better than the one obtained by the mp–method. The second contribution is caused by the fact that the modeling stage can not be optimized with respect to the total design. The case of an arbitrary smoothing lag $m > 0$ will be considered subsequently.

Theorem 3.3 Consider the *multiple prediction method* for designing a DFE with smoothing lag $m = 0$. For a given model S, form the prediction error

$$\begin{aligned}\tilde{y}(t|t-1) &= y(t) - \hat{y}(t|t-1) \\ \hat{y}(t|t-1) &= F_{y,1}(q^{-1})y(t-1) + F_{u,1}(q^{-1})u(t-1)\end{aligned}$$

where the predictor filters $F_{y,1}(q^{-1})$ and $F_{u,1}(q^{-1})$ are designed as optimal linear predictors for the model S. Let the variance of the prediction error be

$$W(S) = \mathrm{E}\ \tilde{y}^2(t|t-1) \tag{3.4.4}$$

for the model S. Then, the optimal model (3.4.1) is given by minimization of the one step ahead prediction error variance

$$\hat{S}_p = \arg\min_{S\in\mathcal{M}} W(S) \quad . \tag{3.4.5}$$

Proof. See Appendix B, Section B.5, where it is also shown that the solution of (3.4.1) leads to an intractable optimization problem for the nominal design ∎

Problem 3.1 presently seems unsolvable for an arbitrary smoothing lag $m > 0$. The reason is that the total performance criterion is not minimized by independently minimizing the variances of multiple prediction errors. The way the prediction errors affect the performance criterion leads, instead, to an intricate optimization problem. For simplicity, we illustrate the problem of one–step smoothing. With the multiple prediction method, the estimate of $u(t-1)$ is obtained as

$$\hat{u}(t-1) = \gamma_2\tilde{y}(t|t-2) + \gamma_1\tilde{y}(t-1|t-2)$$

after minimization, with respect to γ_1 and γ_2, of the MSE criterion

$$V(S_T, [\gamma, S]) = \mathrm{E}[u(t-1) - \hat{u}(t-1)]^2 \quad .$$

The criterion depends on the model S from which the predictions are computed. The prediction errors can be written as

$$\begin{aligned}\tilde{y}(t|t-2) &= [g_0u(t) + g_1u(t-1) + e(t) + h_1e(t-1)] + \epsilon_2(S, t-2) \\ \tilde{y}(t-1|t-2) &= [g_0u(t-1) + e(t-1)] + \epsilon_1(S, t-2)\end{aligned}$$

where g_i and h_i are the impulse response coefficients of the underling system and noise, respectively, and the terms ϵ_i depend on the model S and on past values of $u(t)$ and $e(t)$ up to $t-2$. Introduce the following notation:

$$\begin{aligned}W_i(S) &= \mathrm{E}\ \tilde{y}^2(t-2+i|t-2)\ ,\ i = 1, 2 \\ W_{12}(S) &= \mathrm{E}\ \tilde{y}(t|t-2)\tilde{y}(t-1|t-2) \\ W_\epsilon(S) &= \mathrm{E}\ \epsilon_1(S, t-2)\epsilon_2(S, t-2) \quad .\end{aligned} \tag{3.4.6}$$

After minimization of the criterion with respect to γ, the criterion will only depend on the model S. Straightforwardly from the notation defined above, the criterion value for the optimal γ can be expressed as

$$V(S) = 1 - \frac{g_1^2W_2 + g_0^2W_1 - 2g_0g_1W_{12}}{W_2W_1 + W_{12}^2} \quad . \tag{3.4.7}$$

In general, such an expression can not be minimized with respect to W_1 and W_2 independently, because of the correlation term W_ϵ that appears in the factor W_{12}.

Hence, the model which attains the minimum variance of the prediction errors does in general not solve the Problem 3.1, as was the case with the pure filtering. Minimization of the prediction errors may however still give good results for smoothing lags larger than zero. This property is studied in the following section by means of two examples simple enough to perform exact calculations.

3.5 Examples of Suboptimal Design

This section contains two examples where we illustrate the performance of the nominal design (nom–method) and the multiple prediction method (mp–method) in the case of undermodeling. The two indirect methods are compared to the direct method. Only asymptotic results will be considered, i.e. the identification experiment is supposed to have an infinite duration. The variance contribution will not be considered in the analysis. Hence, the results of the examples illustrate only one of the two main aspects that are needed in order to assess the performance of a design method.

The case $m = 1$ will be considered. Models are obtained by minimizing the variances W_1 and W_2 of the one and two step ahead predictions errors. In the first example, the DFE obtained with the mp–method results in an inferior smoothing performance as compared to the use of a direct method. See Remark 3.5. In the second example a richer model class, that includes predictors with better properties, is considered. In such a case, the design obtained with the mp–method is optimal within the filter class considered. It is not clear whether conditions for optimality of the mp–method within a filter class can be inferred in general cases. This open problem is left for further research.

Example 3.1 Illustration of suboptimal design of DFEs[5].

Data are generated by a system with FIR channel and colored noise

$$\begin{aligned} y(t) &= G_T(q^{-1})u(t) + v(t) \\ \rho_0 &= \mathrm{E}v^2(t) \\ \rho_1 &= \mathrm{E}v(t)v(t-1) \\ G_T(q^{-1}) &= g_0 + g_1 q^{-1} + g_2 q^{-2} + g_3 q^{-3} \end{aligned} \tag{3.5.1}$$

The input $u(t)$ is white and independent of the noise $v(t)$. The system (3.5.1) may represent a digital communication channel with multipath propagation, where only the first impulse response coefficient g_0 corresponds to the direct propagation. The system

[5]The calculations involved in this example are given in Appendix B, Section B.6.

(3.5.1) is modeled by

$$\begin{array}{|rcl|}\hline y(t) &=& B(q^{-1})u(t) + \hat{e}(t) \\ \hat{\rho} &=& \mathrm{E}\hat{e}^2(t) \\ \mathcal{M}_1 &=& \left\{B(q^{-1}) = b_0 + b_1 q^{-1} + b_2 q^{-2}\right\} \\ \hline\end{array} \tag{3.5.2}$$

The model only takes the occurrence of delays up to two sampling periods into account. The correlation of the noise is not modeled.

A DFE design with the indirect methods will have filters with the following structure

$$\begin{array}{|rcl|}\hline F_f(q^{-1}) &=& c_0 + c_1 q^{-1} \\ F_b(q^{-1}) &=& d_0 + d_1 q^{-1} \\ \hline\end{array} \tag{3.5.3}$$

where $F_f(q^{-1})$ and $F_b(q^{-1})$ are the feedforward and the feedback filters, respectively. See Corollary 3.1 and Remark 3.4. In order to simplify the mathematical expressions appearing in the example, the following notation will be used:

$$\begin{aligned}
\mathbf{c} &\triangleq [\, c_0 \quad c_1 \,]^T \;\; ; \;\; \mathbf{d} \triangleq [\, d_1 \quad d_0 \,]^T \\
\mathbf{s} &\triangleq [\, s_0 \quad s_1 \,]^T \;\; ; \;\; \mathbf{l} \triangleq [\, l_1 \quad l_0 \,]^T \\
\gamma &\triangleq [\, \gamma_2 \quad \gamma_1 \,]^T \;\; ; \;\; \mathbf{g} \triangleq [\, g_1 \quad g_0 \,]^T \\
\mathbf{G}_1 &\triangleq \begin{bmatrix} 0 & g_2 \\ g_2 & g_1 \end{bmatrix}^T \;\; ; \;\; \mathbf{G}_2 \triangleq \begin{bmatrix} g_3 & 0 \\ 0 & 0 \end{bmatrix}^T \\
\mathbf{A}_{dir} &\triangleq \begin{bmatrix} \rho_0 + g_0^2 + g_1^2 & \rho_1 + g_0 g_1 \\ \rho_1 + g_0 g_1 & \rho_0 + g_0^2 + g_3^2 \end{bmatrix} \\
\mathbf{A}_n &\triangleq \begin{bmatrix} \rho_0 + g_0^2 + g_1^2 + g_3^2 & g_0 g_1 \\ g_0 g_1 & \rho_0 + g_0^2 + g_3^2 \end{bmatrix} .
\end{aligned}$$

First, the use of a direct method will be considered. Subsequently, the two indirect methods will be applied. The performance of each method will be illustrated by means of a computer simulation at the end of the example.

Direct Method. Given correct past decisions, the estimation error corresponds to

$$z(t) = u(t-1) - F_f(q^{-1})y(t) + F_b(q^{-1})u(t-2) \tag{3.5.4}$$

and the DFE performance is measured by the variance of the estimation error

$$V(F_f, F_b) = \mathrm{E}\; z^2(t) \;\; . \tag{3.5.5}$$

With the direct method, the performance criterion (3.5.5) is directly minimized with respect to the parameters of $F_f(q^{-1})$ and $F_b(q^{-1})$. Observe that the criterion is a quadratic function of the filters parameters. Apparently, this leads to the optimal DFE subjected to the constraint (3.5.3), which can be designed for the system (3.5.1).

The minimization problem can be solved by first minimizing the criterion with respect to the coefficient vector $\mathbf{d}$. This will lead to a new quadratic function of the coefficient vector $\mathbf{c}$, which can be minimized subsequently. The optimal choice for the coefficient vector $\mathbf{d}$ for the feedback filter is given by

$$\hat{\mathbf{d}} = [\mathbf{G}_1 + \mathbf{G}_2]\mathbf{c} \quad . \tag{3.5.6}$$

Then, the criterion (3.5.5) reduces to

$$V(\mathbf{c}) = 1 + \mathbf{c}^T \mathbf{A}_{dir}\mathbf{c} - 2\mathbf{c}^T \mathbf{g} \tag{3.5.7}$$

which can be minimized with respect to the coefficient vector $\mathbf{c}$ for the feedforward filter. The design based on the dir–method is summarized as follows:

$$\boxed{\begin{array}{rcl} \mathbf{A}_{dir}\hat{\mathbf{c}} & = & \mathbf{g} \\ \hat{\mathbf{d}} & = & [\mathbf{G}_1 + \mathbf{G}_2]\hat{\mathbf{c}} \\ V_{dir} & = & 1 - \mathbf{g}^T \mathbf{A}_{dir}^{-1}\mathbf{g} \end{array}} \tag{3.5.8}$$

Indirect Methods. First, a model $\hat{B}(q^{-1})$ has to be obtained in the model class $\mathcal{M}_1$. Consider the model residual, see Lemma 3.1,

$$\hat{e}(t) = y(t) - B(q^{-1})u(t) \quad .$$

From minimization of the variance of the residual, the following model is obtained:

$$\boxed{\begin{array}{rcl} \hat{B}(q^{-1}) & = & g_0 + g_1 q^{-1} + g_2 q^{-2} \\ \hat{\rho} & = & \rho_0 + g_3^2 \end{array}} \tag{3.5.9}$$

Note that bias–free estimates of the first three FIR–channel coefficients are obtained. The unmodeled coefficient g_3 will increase the estimated noise variance. It is straightforward to see that the model (3.5.9) minimizes, over the model class, the variance of both one and two step the prediction errors

$$\begin{array}{rcl} \tilde{y}(t-1|t-2) & = & y(t-1) - (b_1 + b_2 q^{-1})u(t-2) \\ \tilde{y}(t|t-2) & = & y(t) - b_2 u(t-2) \quad . \end{array}$$

On the basis of the model $\hat{B}(q^{-1})$, the DFE is designed with the nominal design and with the multiple prediction method. The matrices $\mathbf{P}$ and $\mathbf{T}$, see Lemma 3.1 and Theorem 3.1, result as

$$\mathbf{P} = \begin{bmatrix} g_0 & 0 \\ g_1 & g_0 \end{bmatrix} \qquad \mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad . \tag{3.5.10}$$

The vectors of coefficients $\mathbf{s}$ for $S_1(q^{-1})$ and $\mathbf{l}$ for $L_{1_*}(q)$ are calculated from the expressions

$$\begin{array}{rcl} \mathbf{s} & = & \mathbf{T}\gamma \\ \mathbf{l} & = & -\mathbf{P}\gamma + \mathbf{r} \end{array}$$

with $\mathbf{r} = [0\ 1]^T$, and with γ to be chosen as in Theorem 3.1 for the nom–method and as in Lemma 3.1 for the mp–method. Then, the DFE filters will be designed from

$$\begin{aligned} q^{-1}Q &= q\hat{B}S_1 + L_{1_*} - 1 \\ F_f &= S_1 \\ F_b &= Q \ . \end{aligned}$$

The above equations result in the following relations for the coefficient vectors for the feedback and feedforward filters:

$$\mathbf{d} = \mathbf{G}_1\gamma \tag{3.5.11}$$
$$\mathbf{c} = \gamma \ . \tag{3.5.12}$$

The indirects methods *will not* find the optimal value of the parameter vector $\mathbf{d}$. Compare (3.5.11) to (3.5.6). Using (3.5.11) and (3.5.12), the criterion (3.5.5) reduces to

$$V(\mathbf{c}) = 1 + \mathbf{c}^T\mathbf{A}_{dir}\mathbf{c} - 2\mathbf{c}^T\mathbf{g} + \mathbf{c}^T\mathbf{G}_2^2\mathbf{c} \ . \tag{3.5.13}$$

The estimation error $V(\mathbf{c})$ will be larger than the value (3.5.7) for the direct method for any value of $\mathbf{c}$, because of the presence of the term $\mathbf{c}^T\mathbf{G}_2^2\mathbf{c} > 0$. Such a difference vanishes when $g_3 = 0$, i.e. when only the noise model is incorrect. With the mp–method, the vector γ (which is equal to $\mathbf{c}$) is chosen by minimizing the MSE criterion (3.5.12). Hence

$$V_{dir} \ \leq \ V_p \ \leq \ V_n \ .$$

The design resulting from the indirect methods can be summarized as follows. The mp–method involves

$$\boxed{\begin{aligned} [\mathbf{A}_{dir} + \mathbf{G}_2^2]\hat{\mathbf{c}} &= \mathbf{g} \\ \hat{\mathbf{d}} &= \mathbf{G}_1\hat{\mathbf{c}} \\ V_p &= 1 - \mathbf{g}^T[\mathbf{A}_{dir} + \mathbf{G}_2^2]^{-1}\mathbf{g} \end{aligned}} \tag{3.5.14}$$

The nom–method can be expressed as

$$\boxed{\begin{aligned} \mathbf{A}_n\hat{\mathbf{c}} &= \mathbf{g} \\ \hat{\mathbf{d}} &= \mathbf{G}_1\hat{\mathbf{c}} \\ V_n &= 1 + \mathbf{g}^T\mathbf{A}_n^{-1}[\mathbf{A}_{dir} + \mathbf{G}_2^2]^{-1}\mathbf{A}_n^{-1}\mathbf{g} - 2\mathbf{g}^T\mathbf{A}_n^{-1}\mathbf{g} \end{aligned}} \tag{3.5.15}$$

Discussion. In order to quantify the loss of performance of the indirect methods, an experiment has been carried out based on numerical simulations. $N_{\text{sys}} = 1000$ true channels with structure:

$$G_{T1}(q^{-1}) = g_0 + g_1q^{-1} + g_2q^{-2}$$

were randomly generated, with $g_i \in \mathcal{N}(0,1)$, $i = 0, 1, 2$. To each channel, a third term

$$g_3q^{-3}$$

Figure 3.6: MSE performance of DFEs designed in Example 3.1 with direct and indirect methods, averaged over 1000 randomly selected FIR channels. Solid: dir–method. Dashed: mp–method. Dashed–dotted: nom–method. Correlated noise, with $\rho_1 = r\rho_0$. SNR = 10dB.

was added, with

$$g_3^2 = \alpha g_2^2 \quad .$$

The parameter α varies between 0 and 0.4. The term g_3 may represent an unmodeled multipath propagation with long delay in a digital communication channel. The signal to noise ratio was set to SNR = 10dB at the output of the system G_{T1} (hence, when g_3 increases in magnitude, the actual signal to noise ratio is improved). The DFE was designed for each system according to the methods above, and the performance was averaged for a given value of α over the 1000 systems. In Figure 3.6 the averaged performance is shown, for different values of ρ_1. In the upper left–hand figure, the measurement noise is white ($\rho_1 = 0$). Hence, the performance differences are caused only by the system undermodeling. Note how these differences are modified when increasing the correlation factor, i.e adding undermodeling in the noise spectrum. The dir–method shows to be rather robust, since it does not rely on any modeling assumption and fully adapts to the unknown underlying environment. Such robustness is lost when using the nom–method, which bases the design completely on the (incorrect) model. The mp–method represents a compromise between the two, since both indirect modeling (the predictors estimation) and direct adaptation (optimization with respect to γ) are utilized ■

On the basis of Example 3.1, it seems that the indirect methods fail to provide the best DFE design for the given filter structure. However, this is not always the case, as shown by the following example.

Example 3.2 Optimality of the multiple prediction method[6].

For the same true system in Example 3.1, in this example it is shown that the smoothing criterion can be optimized with respect to the choice of the predictors to be used with the multiple prediction method, if a model class richer than $\mathcal{M}_1$ in (3.5.2) is considered. The resulting DFE is the optimal DFE that can be designed with the filter structure imposed by the model class. Hence, the expression (3.4.3) holds. The key point is that the expression (3.4.7) can, in this example, be minimized independently with respect to the variances W_1 and W_2 of the one and two step ahead predictions errors.

Data are generated from the system (3.5.1) as in Example 3.1. The system is now modeled as

$$\begin{aligned} y(t) &= B(q)u(t) + \frac{1}{N(q)}\hat{e}(t) \\ \hat{\rho} &= \mathrm{E}\hat{e}^2(t) \\ \mathcal{M}_2 &= \left\{B(q) = b_0 + b_1 q^{-1} + b_2 q^{-2},\ N(q) = 1 + n_1 q^{-1}\right\} \end{aligned} \tag{3.5.16}$$

The difference as compared to the model class $\mathcal{M}_1$ considered in Example 3.1 is that the correlation of the noise is also modeled. An autoregressive model is utilized.

For the noise in the system (3.5.1), assume

$$\mathrm{E}v(t)v(t-2) = \rho_2 > 0 \quad .$$

Note that the results of Example 3.1 were independent of the correlation ρ_2.

A DFE design with the mp–method will have filters with the following structure

$$\begin{aligned} F_f(q^{-1}) &= c_0 + c_1 q^{-1} + c_2 q^{-2} \\ F_b(q^{-1}) &= d_0 + d_1 q^{-1} + d_2 q^{-2} \end{aligned} \tag{3.5.17}$$

where $F_f(q^{-1})$ and $F_b(q^{-1})$ are the feedforward and the feedback filters, respectively. See Corollary 3.1.

First, the use of a direct method will be considered, in order to assess the optimal performance attainable with the filter structures (3.5.17). Subsequently, the mp–method will be applied. It will be shown that the resulting DFE attains the same performance. It is worth noticing that by minimizing the variances W_1 and W_2 of the one and two step ahead prediction errors independently, two different models in $\mathcal{M}_2$ are obtained. Thus, the DFE filters will not have the structure given by Corollary 3.1, where it was assumed that all the predictors were calculated from the same model. Instead, they will have the more general structure depicted in Figure 3.5.

[6]The calculations involved in the example are given in Appendix B, Section B.7.

In order to simplify the mathematical expressions appearing in the example, the following notation will be used:

$$
\begin{aligned}
\mathbf{c} &\triangleq [\ c_0 \quad c_1 \quad c2\ ]^T \ ; \ \mathbf{d} \triangleq [\ d_2 \quad d_1 \quad d_0\ ]^T \\
\gamma &\triangleq [\ \gamma_2 \quad \gamma_1\ ]^T \\
\mathbf{g} &\triangleq [\ g_1 \quad g_0\ ]^T \ ; \ \mathbf{g}_1 \triangleq [\ \mathbf{g} \quad 0\ ]^T \\
\mathbf{G} &\triangleq \begin{bmatrix} 0 & g_3 & g_2 \\ g_3 & g_2 & g_1 \\ g_2 & g_1 & g_0 \end{bmatrix}^T \\
\mathbf{A}_{dir} &\triangleq \begin{bmatrix} \rho_0 + g_0^2 + g_1^2 & \rho_1 + g_0 g_1 & \rho_2 \\ \rho_1 + g_0 g_1 & \rho_0 + g_0^2 & \rho_1 \\ \rho_2 & \rho_1 & \rho_0 + g_3^2 \end{bmatrix} .
\end{aligned}
$$

Direct Method. With the direct method, the performance criterion (3.5.5) is directly minimized with respect to the parameters of $F_f(q^{-1})$ and $F_b(q^{-1})$. As in Example 3.1, the minimization is performed by first minimizing the criterion with respect to the coefficient vector $\mathbf{d}$. This will lead to a new quadratic function of the coefficient vector $\mathbf{c}$, which can be minimized subsequently. The optimal choice for the coefficient vector $\mathbf{d}$ for the feedback filter is given by

$$\hat{\mathbf{d}} = \mathbf{G}\mathbf{c} \ . \tag{3.5.18}$$

Then, the criterion (3.5.5) reduces to

$$V(\mathbf{c}) = 1 + \mathbf{c}^T \mathbf{A}_{dir} \mathbf{c} - 2\mathbf{c}^T \mathbf{g}_1 \tag{3.5.19}$$

which can be minimized with respect to the coefficient vector $\mathbf{c}$ for the feedforward filter. The design based on the dir–method is summarized as follows:

$$
\boxed{
\begin{aligned}
\mathbf{A}_{dir}\hat{\mathbf{c}} &= \mathbf{g}_1 \\
\hat{\mathbf{d}} &= \mathbf{G}\hat{\mathbf{c}} \\
V_{dir} &= 1 - \mathbf{g}_1^T \mathbf{A}_{dir}^{-1} \mathbf{g}_1
\end{aligned}
} \tag{3.5.20}
$$

Multiple Prediction Method. First, a model

$$S = \{\hat{B}, \hat{N}, \hat{\rho}\}$$

has to be obtained in the model class $\mathcal{M}_2$ in order to compute the predictors filters. Then the estimate of $u(t-m)$ is obtained as

$$\hat{u}(t-1) = \gamma_2 \tilde{y}(t|t-2) + \gamma_1 \tilde{y}(t-1|t-2) \tag{3.5.21}$$

after minimization, with respect to γ_1 and γ_2, of the MSE criterion

$$V(S_T, [\gamma, S]) = \mathrm{E}[u(t-1) - \hat{u}(t-1)]^2 \ . \tag{3.5.22}$$

Denote the variance of the one and two step ahead prediction errors as in (3.4.6):

$$W_i(S) = \mathrm{E}\ \tilde{y}^2(t-2+i|t-2) \ , \ i = 1, 2 \ . \tag{3.5.23}$$

The minimization of (3.5.23) with respect to the model S will be performed independently for $i = 1, 2$. Hence, two different models S_1 and S_2 will result in general.

For both $i = 1, 2$, the predictor filters $F_{y,i}$ and $F_{u,i}$ designed as being optimal for the models in $\mathcal{M}_2$ will have the following structure, see Appendix B, Section B.7:

$$F_{y,i}(q^{-1}) = m \tag{3.5.24}$$
$$F_{u,i}(q^{-1}) = r_0 + r_1 q^{-1} + r_2 q^{-2} \ . \tag{3.5.25}$$

The optimal predictor filters are found as

$$\hat{F}_{y,1}(q^{-1}) = \hat{m}_1 = \frac{\rho_1}{\rho_0 + g_3^2} \tag{3.5.26}$$
$$\hat{F}_{y,2}(q^{-1}) = \hat{m}_2 = \frac{\rho_2}{\rho_0 + g_3^2} \tag{3.5.27}$$
$$\hat{F}_{u,1}(q^{-1}) = (g_1 - \hat{m}_1 g_0) + (g_2 - \hat{m}_1 g_1)q^{-1} + (g_3 - \hat{m}_1 g_2)q^{-2} \tag{3.5.28}$$
$$\hat{F}_{u,2}(q^{-1}) = (g_2 - \hat{m}_2 g_0) + (g_3 - \hat{m}_2 g_1)q^{-1} - \hat{m}_2 g_2 q^{-2} \ . \tag{3.5.29}$$

The feedforward and the feedback filters will have the following structure, see Figure 3.5:

$$\begin{aligned} F_f(q^{-1}) &= c_0 + c_1 q^{-1} + c_2 q^{-2} \\ &= \gamma_2 + \gamma_1 q^{-1} - \left[\gamma_1 F_{y,1}(q^{-1}) + \gamma_2 F_{y,2}(q^{-1})\right] q^{-2} \end{aligned} \tag{3.5.30}$$
$$\begin{aligned} F_b(q^{-1}) &= d_0 + d_1 q^{-1} + d_2 q^{-2} \\ &= \gamma_1 F_{u,1}(q^{-1}) + \gamma_2 F_{u,2}(q^{-1}) \ . \end{aligned} \tag{3.5.31}$$

Introduce the following matrix:

$$\mathbf{B}_p \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\hat{m}_2 & -\hat{m}_1 \end{bmatrix} \ . \tag{3.5.32}$$

With the use of (3.5.32), from (3.5.30) and (3.5.31), the vectors of coefficients $\mathbf{d}$ and $\mathbf{c}$ are obtained as

$$\mathbf{c} = \mathbf{B}_p \gamma \tag{3.5.33}$$
$$\mathbf{d} = \mathbf{G}\mathbf{B}_p \gamma \ . \tag{3.5.34}$$

Expression (3.5.34) coincides with (3.5.18). The MSE criterion is thus optimized with respect to the vector of coefficients $\mathbf{d}$, and reduces to the expression (3.5.19), with the vector of coefficients $\mathbf{c}$ constrained by (3.5.33). The MSE criterion is then given by

$$V(\gamma) = 1 + \gamma^T \mathbf{B}_p^T \mathbf{A}_{dir} \mathbf{B}_p \gamma - 2\gamma^T \mathbf{B}_p^T \mathbf{g}_1 \ . \tag{3.5.35}$$

Observe that the following holds:

$$\mathbf{B}_p^T \mathbf{g}_1 = \mathbf{g} \ .$$

After minimization of the expression (3.5.35) with respect to the vector of coefficients γ, the smoothing performance of the mp–method is obtained as

$$\boxed{V_p = 1 - \mathbf{g}^T [\mathbf{B}_p^T \mathbf{A}_{dir} \mathbf{B}_p]^{-1} \mathbf{g}} \tag{3.5.36}$$

In Appendix B, Section B.7, it is shown that the value given in (3.5.36) coincides with the global minimum (3.5.20) of the the performance criterion (3.5.5) over the set of filters with structure (3.5.17). The mp–method thus results in the best design for the given filter structures ■

Chapter 4

Some Issues on System Identification

4.1 Introduction

In this chapter, the sample behavior of direct and indirect methods will be analysed, and some issues concerning the use of system identification will be addressed.

In Chapter 1, two basic approaches for data–based design were distinguished, namely, *direct* and *indirect* methods. Indirect methods based on approximate models have been analyzed in Chapter 2, for the linear deconvolution filter, and in Chapter 3, for the decision feedback equalizer. The analysis was conducted on the basis of asymptotic considerations, as if the identification experiment had an infinitely long duration. In other words, bias errors, but not *variance errors*, were taken into account. The variance error will also have to be considered in order to assess the quality of a certain design method. Under the assumption that the system generating the data is time–invariant, the variance error measures the sensitivity of an estimated model to the specific realization of the noise during a particular identification experiment.

In Section 1.4, the direct method was shown to have the best asymptotic behavior. If the number of data available for the design is large enough, a filter designed with the direct method will, in principle, be the optimal design attainable for a given filter structure. The utility of direct methods can, however, be reduced in situations with short data sequences, if the design outcome is sensitive to the noise realization. The above issue will be analyzed in Section 4.2. It will be argued, and shown by an example, that the direct method does, in fact, show a pronounced sensitivity. This sensitivity can result in designs with a very poor performance.

In Section 4.3, the sample behavior of indirect methods will be analyzed, and compared to that of direct methods. It will be argued that indirect methods will results in less sensitive designs. General considerations concerning the estimation of models for indirect methods will also be given. The conclusions inferred from the analysis of the sample behavior will be investigated by simulations in Chapter 5.

The experiments in Chapter 5 will be conducted under the assumption of (partly) known initial conditions. That assumption may not be fulfilled in practice. In Section 4.4, we will investigate the effect of unknown initial conditions on the filter design in situations with short data sequences.

In both direct and indirect methods, a model or a filter structure must be selected. Section 4.5 will give a brief overview of that problem, which goes under the name of model validation or structure selection.

4.2 The Direct Method

Direct methods for filter design were introduced in Section 1.4. In this section, the sample behavior of direct methods will be analyzed. Our main interest will be to investigate the effect that the noise realization during the identification experiment has on the filter estimate.

The notation introduced in Section 1.3 is repeated here for the convenience of the reader. Data are assumed to be generated by the linear, time–invariant and stable system

$$\begin{aligned} y(t) &= G_T(q^{-1})u(t) + v(t) \\ G_T(q^{-1}) &= \sum_{k=0}^{\infty} g_k q^{-k} \qquad (4.2.1) \\ \mathrm{E}v^2(t) = \lambda_v \ &; \ \mathrm{E}u^2(t) = 1 \ . \end{aligned}$$

The input process $u(t)$ is assumed to be white and, without loss of generality, to have unit variance. The measurement noise $v(t)$ is stationary, with an arbitrary spectrum. The input and the noise are assumed to be mutually uncorrelated. The input and output sequences are available in the interval $t = 1, \ldots, N_d$. The effect of unknown initial conditions is assumed to be negligible. See Section 4.4. Denote the noise–free component of the output signal $y(t)$ by

$$y_{nf}(t) = \sum_{0}^{t-1} g_k u(t-k) \ .$$

First, we shall consider the estimation of a linear deconvolution filter, see Chapter 2. The estimation of the filters in a decision feedback equalizer, see Chapter 3, will be considered subsequently.

Assume that the structure of the filter $F_{dir}(q^{-1})$ to be designed with the direct method has been fixed by the choice of a filter class $\mathcal{F}$. (The problem of how to choose a filter class will be addressed in Section 4.5.) In Section 1.4, it was shown that the filter $F_{dir}(q^{-1})$ is obtained from the model equation

$$u(t-m) = F(q^{-1})y(t) + z(t) \qquad (4.2.2)$$

$$W_{id}(F) = \frac{1}{N_d} \sum_{t=m+1}^{N_d} z^2(t) \qquad (4.2.3)$$

by solving

$$F_{dir}(q^{-1}) = \arg\min_{\mathcal{F}} W_{id}(F) \quad . \tag{4.2.4}$$

In the above expressions, m is the smoothing lag, $z(t)$ is the estimation error when the input estimate is obtained by the filter $F(q^{-1})$, and $W_{id}(F)$ is the loss function of the identification algorithm, which coincides with the sample performance on the training data. In principle, the minimization problem in (4.2.4) can be solved. We will not be concerned with problems related to local minima of the criterion.

It is a standard result that, asymptotically in the number of data, the parameter estimate obtained by (4.2.2)–(4.2.4) will converge to a Gaussian distributed stochastic variable under mild conditions, ([SödSt89], p.210). The mean value of the estimate will coincide with the parameter values of the optimal filter in $\mathcal{F}$, see Section 1.4. The variance of the estimate can also be evaluated. It will, in general, depend on the noise correlation, on the system transfer function and on the filter structure. Except from simple cases, it is not possible to obtain closed form expressions for the asymptotic variance. If the number of data N_d is small, it is very difficult to assess the variability of the estimate. However, by qualitatively analyzing how the estimate (4.2.4) is obtained, some meaningful conclusions can be drawn.

Denote the impulse response coefficients of a filter $F(q^{-1})$ by

$$F(q^{-1}) = \sum_{0}^{\infty} f_k q^{-k} \quad .$$

With the use of the introduced notations, the loss function (4.2.3) can then be rewritten as:

$$W_{id}(F) = \frac{1}{N_d} \sum_{t=1+m}^{N_d} \left[u(t-m) - \sum_{k=0}^{t-1} f_k y_{nf}(t-k) - \sum_{k=0}^{t-1} f_k v(t-k) \right]^2 \quad . \tag{4.2.5}$$

The minimization in (4.2.4) is performed with respect to the impulse response coefficients $\{f_i\}_0^{N_d-1}$, subjected to the constraint $F(q^{-1}) \in \mathcal{F}$. In (4.2.5), the summations in the index k can be extended up to the index $k = N_d - 1$ by assuming all signals to be zero before $t = 1$.

$$\begin{aligned} W_{id}(F) &= \\ &= \sum_{k=0}^{N_d-1} \sum_{j=0}^{N_d-1} f_k f_j \left[\frac{1}{N_d} \sum_{t=1+m}^{N_d} v(t-k)v(t-j) + \frac{1}{N_d} \sum_{t=1+m}^{N_d} y_{nf}(t-k) y_{nf}(t-j) \right. \\ &\left. + \frac{2}{N_d} \sum_{t=1+m}^{N_d} y_{nf}(t-k)v(t-j) \right] + \frac{1}{N_d} \sum_{t=1+m}^{N_d} u^2(t-m) \\ &-2 \sum_{k=0}^{N_d-1} f_k \left[\frac{1}{N_d} \sum_{t=1+m}^{N_d} u(t-m) y_{nf}(t-k) + \frac{1}{N_d} \sum_{t=1+m}^{N_d} u(t-m) v(t-k) \right] \end{aligned} \tag{4.2.6}$$

For ergodic processes $s_i(t)$ that are zero for $t \leq 0$, consider the following sample pseudo–correlation coefficients:

$$\hat{r}_s(k, j, N_d) \quad = \quad \frac{1}{N_d} \sum_{t=1+m}^{N_d} s(t-k)s(t-j)$$

$$\hat{r}_{s_1,s_2}(k,j,N_d) \;=\; \frac{1}{N_d}\sum_{t=1+m}^{N_d} s_1(t-k)s_2(t-j)$$

Note that the pseudo–correlation coefficients depend on both indexes k, j, not only on their difference. Asymptotically in the number of data, the sample pseudo–correlation coefficients will coincide with well defined correlation sequences of the corresponding processes. For ease of notation, the index N_d will be omitted.

The loss function (4.2.6) can be rewritten as the sum of three terms

$$W_{id}(F) = W_u(F) + W_{oe}(F) + W_v(F) \tag{4.2.7}$$

where

$$W_u(F) \;=\; \hat{r}_u(0,0) + \sum_{k=0}^{N_d-1}\sum_{j=0}^{N_d-1} f_k f_j \hat{r}_{y_{nf}}(k,j) - 2\sum_{k=0}^{N_d-1} f_k \hat{r}_{u,y_{nf}}(m,k) \tag{4.2.8}$$

$$W_{oe}(F) \;=\; -2\sum_{k=0}^{N_d-1} f_k \hat{r}_{u,v}(m,k) \tag{4.2.9}$$

$$W_v(F) \;=\; \sum_{k=0}^{N_d-1}\sum_{j=0}^{N_d-1} f_k f_j \left[\hat{r}_v(k,j) + 2\hat{r}_{y_{nf},v}(k,j)\right] \tag{4.2.10}$$

The term $W_u(F)$ is the part of the loss function which is independent of the noise realization. The other two parts have been distinguished for further comparison to the analysis of indirect methods in the next section.

The loss function (4.2.7) should be compared to the asymptotic loss function. Assume the number of data N_d to be large enough to consider the contribution of the "tail", f_i for $i > N_d$, on the asymptotic loss function negligible[1]. Then the asymptotic loss function will differ from (4.2.7) only due to the use of pseudo–correlation coefficients in (4.2.7). The latter will thus result in the estimate (4.2.4) to be different from the optimal filter in $\mathcal{F}$.

An indication of the sensitivity of the estimate (4.2.4) to the noise realization can be given by studying the variations of the terms that weight the impulse response coefficients of the filter $F(q^{-1})$. This does, of course, not completely assess the sensitivity. It rather serves as a qualitative indication. The quadratic terms $f_k f_i$ are weighed by the coefficients

$$A_{k,j} \triangleq \hat{r}_{y_{nf}}(k,j) + 2\hat{r}_{y_{nf},v}(k,j) + \hat{r}_v(k,j) \tag{4.2.11}$$

while the linear terms f_k are weighed by

$$B_k \triangleq \hat{r}_{u,y_{nf}}(m,k) + \hat{r}_{u,v}(m,k) \quad . \tag{4.2.12}$$

The weights $A_{k,j}$ are affected by the individual noise realization via the two last terms in (4.2.11). The weights B_k are affected via the last term in (4.2.12). Let us in more detail consider the weights B_k in (4.2.12), and set $m = 0$ (filtering case). Then the pseudo–correlation coefficients $\hat{r}_{u,y_{nf}}(0,k)$ measure the correlation of the present input

[1] The impulse response coefficients f_i of a stable system decay, for i large enough, proportionally to ρ_D^i, where ρ_D is the magnitude of the system pole closest to the unit circle.

value $u(t)$ with past noise–free output values $y_{nf}(t-k)$. Since the input is white, the coefficients will vary randomly around zero for $k > 0$, and around the value g_0 of the first impulse response coefficient for $k = 0$. The coefficients $\hat{r}_{u,v}(0,k)$ will also vary around zero, and the influence of the noise can be expected to be *high* also for the weights B_k. In the case of smoothing lag $m > 0$, an increasing number of coefficients $\hat{r}_{u,y_{nf}}(m,k)$ will have a non–zero mean, and the influence of the noise can be considered reduced. The qualitative conclusions on the noise sensitivity of the direct method can be then summarized as follows:

1. The terms quadratic in the variables f_i are explicitly influenced by the specific noise realization, via both its pseudo auto–correlation and the pseudo cross–correlation with the output signal.
2. The terms linear in the variables f_i are influenced by the specific noise realization via the pseudo cross–correlation with the input signal. The influence on the weights may be high. Such influence may be reduced by increasing the smoothing lag.

Note that it is not surprising to expect a high sensitivity to the particular noise realization, since the direct method tries to cancel the actual noise effect to obtain a good match between the filter output and the measured input data.

The same reasoning as above could, in principle, be repeated for the estimation of the filters in the feedback equalizer. In Section 1.4, it was shown that the DFE filters can be estimated in the model equation

$$\left[1 + q^{-1}F_b(q^{-1})\right] u(t-m) = F_f(q^{-1})y(t) + z(t) \tag{4.2.13}$$

where $F_f(q^{-1})$ and $F_b(q^{-1})$ are the feedforward and feedback filters, respectively. Note that the model equation (4.2.13) corresponds to the model equation (4.2.2) with a colored input

$$\bar{u}(t-m) \triangleq \left[1 - q^{-1}F_b(q^{-1})\right] u(t-m) \quad .$$

The analysis is much more intricate for the DFE than for the linear deconvolution estimator, due to the coupled estimation of the impulse response coefficients in both filters. The pseudo correlations coefficients in (4.2.11) and (4.2.12) will now depend on the impulse response coefficients of the feedback filter $F_b(q^{-1})$. The specific noise realization will still affect both coefficients $A_{k,j}$ and B_k in (4.2.11) and (4.2.12), and its effect is expected to be similar to that on the design of linear deconvolution filters.

In the example below we will illustrate the sensitivity of the direct estimation of linear deconvolution filters by means of a computer simulation. The main effect is the possible occurrence of cases with very poor performance, *outliers*. The observation of such phenomena seems to confirm the analysis above, where the estimate was expected to vary in an uncontrolled way depending on the pseudo correlation of the noise sequence.

Example 4.1 Sensitivity of the direct method. Data were generated by a "nice" system chosen in the set S_{ni}. The set of channel and noise configurations S_{ni}, already introduced in Example 2.1 in Chapter 2, will be described in Section 5.4. The noise

	filter order					
	3	4	5	6	7	8
mean	0.047	0.060	0.086	0.180	0.150	0.228
std	0.010	0.029	0.064	0.205	0.169	0.322
out	0	1	2	2	1	2
$2 \cdot V_W$	10	8	5	3	4	3

Table 4.1: Example 4.1: sensitivity of the direct method, with a filtering estimator. Channel in $\mathcal{S}_{ni}$ with white noise. SNR = 13dB. Optimal performance: $V_W = 0.037$. Number of designs with different input and noise realizations: 10. First row: mean performance (after elimination of outliers). Second row: standard deviation. Third row: number of cases with performance worse than 1. Fourth row: number of cases with performance better than $2 \cdot V_W$. DATA = 50.

was white and stationary with SNR = 13dB. Filtering estimators of different orders were designed with the direct method for 10 different input and noise realizations of 50 data. The input signal was white, with unit variance, and it was set to zero before the identification experiment. The noise was stationary. The optimal attainable performance was

$$V_W = 0.037$$

A statistics of the performance attained by the designs over the 10 realizations is shown in Table 4.1. In the fourth row, corresponding to the label "out", the number of outliers is reported, where an outlier is defined as a filter that results in a performance

$$V(S_T, F) \ \geq \ 1 \ \ .$$

A performance equal to 1 is attained by the trivial estimate $\hat{u}(t) = 0$. After elimination of the outliers, the average performance and its standard deviation were noted. The values are reported in the first two rows of the tables. In the last row, the number of case in which the filter resulted in a performance

$$V(S_T, F) \ \leq \ 2V_W$$

is also reported. The direct method reveals a pronounced sensitivity to the noise realization, which results in design cases with a very poor performance. The number of estimated parameters in a successful design is severely limited by this sensitivity. The above phenomena will be confirmed by the simulation studies in Chapter 5, where a larger statistics of design cases will be considered. It is worth noticing that indirect methods will (almost) never result in outliers cases ■

4.3 Estimating Models for Indirect Methods

In this section we will consider models used in indirect methods for filter design. Subsequently, the qualitative sensitivity analysis of the previous section will be repeated for an indirect method. It will be argued that the considered indirect method is expected to be less sensitive to noise realizations than the direct method. That conclusion will be confirmed by the simulation studies in Chapter 5.

Strategies for obtaining approximate models that optimize the global design for indirect methods, for a given filter structure, have been analyzed in Sections 2.4 and 3.4. In Section 2.4, it was shown that for the design of linear deconvolution filters there are no clear guidelines for obtaining models. We proposed a two–stage indirect method based on attaining a suitable distribution in the frequency domain of the bias in the channel model. In Section 3.4, it was shown that a multi–predictor approach that minimizes the variance of output predictions error can improve the design of decision feedback equalizers. The strategy always results in the optimal design in the case of filtering.

The results above rely on asymptotic considerations, and their utility may be reduced in situations with short data sequences available for estimation.

Indirect methods require a parametric model of both the channel transfer function and the noise spectrum. The models may be obtained by using two alternative strategies

One–Step methods: channel and noise models are obtained simultaneously, as, for instance, in a prediction error method (PEM).

Two–Step methods: first a model for the channel transfer function is obtained without modeling the noise. Then, as a second step, the noise model is obtained from the resulting residual sequence.

In the literature, there are no general recipes of what strategy or specific method should be used. It is reasonable to argue that the second approach may be preferred in situations with short data sequences. First, it is well known that is somewhat easier to obtain a good model for the channel, for which an input is available, than for the noise spectrum. It is thus desirable to maintain a distinction between the two estimations. Second, a smaller number of parameters are estimated on the available data as compared to the former approach. Third, control of the bias of the channel model is simpler with two–step methods than with one–step methods. Fourth, the model validation stage may be simplified, because a smaller number of model structures can be considered, see Section 4.5.

Classifications and lists of identification methods can be found in any textbook on system identification. In our study we only considered average–case identification based on stochastic assumptions. See Section 1.3. In such an estimation framework, Ljung distinguishes between two different approaches to estimation, namely prediction error based methods and correlation based methods, ([Lju87], Chapter 7). The correlation based methods, as, for instance, the Instrumental Variables methods, are based on demanding non correlation between prediction errors and past data. If the amount of data is small, there are reasons to believe that results related to computing correlations may be rather sensitive to the specific data realization. On the other hand, correlation based methods avoid minimizations of loss functions, and may be preferred from a computational point of view. We will not consider correlation based methods. The analysis of their utility is left for further research on the subject.

On the basis of the considerations above, we elected as the *standard model estimation method* a two–step method where the channel model is estimated with an output–error method at the first step, and the noise model is estimated at the second step from the resulting residual with a prediction error method applied to an ARMA structure. The use of such a method is suitable for indirect design of linear deconvolution filters. Its use will also be investigated for the design of DFEs. See Chapter 5.

Next, we shall repeat the qualitative sensitivity analysis of the previous section for the design of linear deconvolution filters with indirect methods. The indirect method considered is a nominal design based on models obtained with the standard model estimation method. In Section 2.4, it was argued that in cases in which a good deconvolution performance can be attained, the optimal filter can be expected to be primarily determined by the channel inversion. If the noise level is reasonably estimated, the channel model will thus determine the design more than the noise model does.

Data are assumed to be generated by the system (4.2.1). The same assumptions as in Section 4.2 hold. The model of the channel transfer function $G_T(q)$ is estimated from the model equation

$$y(t) = G(q^{-1})u(t) + \hat{v}(t) \tag{4.3.1}$$

$$W_{id}(G) = \frac{1}{N_d} \sum_{t=m+1}^{N_d} \hat{v}^2(t) \tag{4.3.2}$$

by solving

$$G_{oe}(q^{-1}) = \arg\min_{\mathcal{G}} W_{id}(G) \tag{4.3.3}$$

where $\mathcal{G}$ is a model class assumed to be given. In principle, the minimization problem in (4.3.3) can be solved. We will not be concerned with problems related to local minima of the criterion. Introduce the following notation,

$$G(q^{-1}) = \sum_0^\infty \hat{g}_k q^{-k} \quad .$$

Repeating the same steps as in the previous section, and using the same notations there introduced, the loss function $W_{id}(G)$ (4.3.2) can be written as the sum of three terms

$$W_{id}(G) = W_u(G) + W_{oe}(G) + W_v(G) \tag{4.3.4}$$

where

$$W_u = \hat{r}_{y_{nf}}(0,0) + \sum_{k=0}^{N_d-1} \sum_{j=0}^{N_d-1} \hat{g}_k \hat{g}_j \hat{r}_u(k,j) - 2 \sum_{k=0}^{N_d-1} \hat{g}_k \hat{r}_{y_{nf},u}(0,k) \tag{4.3.5}$$

$$W_{oe} = -2 \sum_{k=0}^{N_d-1} \hat{g}_k \hat{r}_{v,u}(0,k) \tag{4.3.6}$$

$$W_v = \hat{r}_v(0,0) + 2\hat{r}_{y_{nf},v}(0,0) \quad . \tag{4.3.7}$$

The loss function (4.3.4) is minimized with respect to the impulse response coefficients $\{\hat{g}_i\}_0^{N_d-1}$, subjected to the constraint $G(q^{-1}) \in \mathcal{G}$. Note that those variable do not affect the noise term W_v.

The expressions above should be compared to (4.2.8)–(4.2.10). The quadratic terms $\hat{g}_k\hat{g}_i$ are weighted by the coefficients

$$\bar{A}_{k,j} \triangleq \hat{r}_u(k,j) \tag{4.3.8}$$

while the linear terms $\hat{g}_i$ are weighted by

$$\bar{B}_k \triangleq \hat{r}_{y_{nf},u}(0,k) + \hat{r}_{v,u}(0,k) \quad . \tag{4.3.9}$$

Compare to (4.2.11) and (4.2.12). The weights $\bar{A}_{k,j}$ in (4.3.8) are not affected by the noise realization as was the case for the direct method. The noise still influences the weights $\bar{B}_k$ in (4.3.9) via the cross pseudo–correlation with past input values. The noise influence is added to the cross pseudo–correlation coefficients $\hat{r}_{y_{nf},u}(0,k)$, that measure the correlation between the present noise–free output value $y_{nf}(t)$ and past input values $u(t-k)$. Those coefficients will vary randomly around the corresponding impulse response coefficient g_k of the channel. They can be considered larger in magnitude than the quantities involving the noise contribution. The noise sensitivity is thus expected to be reduced also for the weights $\bar{B}_k$, as compared to the weights B_k in (4.2.12).

As compared to the direct method, there are thus two different reasons for the reduction of sensitivity in the channel model. The further contribution on the sensitivity due to the estimation of the noise model will not significantly modify the nominal design, in cases when a linear filter is a reasonable solution to the deconvolution problem. It should be finally recalled that the nonlinear transformation of the model parameters into the filter parameters determined by the design equations will further modify the model sensitivity. That effect is very difficult to quantify. However, the simulation studies in Chapter 5 indicate that the good properties of the channel model are maintained by the global design.

4.4 The Problem of Unknown Initial Conditions

Unknown initial conditions constitute a source of error when estimating a model from finite data sequences. In order to better assess the effect of undermodeling, in the simulation studies in Chapter 5 the possible error due to unknown initial conditions will be reduced by assuming that the input signal was not exciting the system before the identification experiment. In this section we will investigate the effect of applying an unknown input, different from zero, on the system before the identification experiment.

Assume that data are generated by the system (4.2.1), and that the input and output sequences are available for $t = 1, \ldots, N_d$. Consider the use of an output–error method for estimating a model $G(q^{-1})$ of the channel $G_T(q^{-1})$. The model is obtained from the model equation

$$y(t) = G(q^{-1})u(t) + \hat{v}(t) \tag{4.4.1}$$

$$G(q^{-1}) = \frac{B(q^{-1})}{A(q^{-1})} \tag{4.4.2}$$

after minimization of the loss–function

$$W_{id}(G) = \frac{1}{N_d}\sum_{t=1}^{N_d}\hat{v}^2(t) \tag{4.4.3}$$

within some model class $\mathcal{G}$. From (4.4.1), the model residual $\hat{v}(t)$ (or output error) in (4.4.3) is computed from the equations

$$\begin{aligned} \hat{v}(t) &= y(t) - \hat{y}(t) \\ A(q^{-1})\hat{y}(t) &= B(q^{-1})u(t) \quad . \end{aligned} \tag{4.4.4}$$

The difference equation (4.4.4) depends on values of the input and output signals before the observation period $t = 1, \ldots, N_d$, and must be initialized. Observe that for the initialization,

$$n = \max(nb, na)$$

initial conditions, where nb and na are the polynomial degrees of the model transfer function, are required. If the n initial conditions are unknown, it is common practice to set them to arbitrary values. For instance, in the routines available in the System Identification Toolbox of Matlab, [Matlab], the initial conditions of (4.4.4) are set so that the model residual $\hat{v}(t)$ is zero for $t = 1, \ldots, n$. An alternative is to set the initial conditions to zero. Of course, if the system was in a different initial condition, the above choices will constitute a source of model error. Since the effect of the initial conditions decay at a rate determined by the slowest pole in the difference equation (4.4.4), such an effect can be considered negligible if the number of data is large enough. Another strategy is to discard the first data up to $t = n$, and to use them as the (correct) initial conditions. For model structures more general than (4.4.1), correct initial conditions can not be recovered by discarding data.

If the initial conditions are unknown, they can, in principle, be estimated. In ([SödSt89], p.490), it is suggested to include them in the parameter vector. The identification algorithm will have to be modified for the computation of the gradient of the loss function (4.4.3) with respect to the extended parameter vector

$$\theta_1 = \begin{bmatrix} \theta & \theta_{ic} \end{bmatrix}^T$$

where θ is the vector formed with the model parameters and θ_{ic} the vector formed with n unknown initial conditions. However, it is important to recall that the initial conditions can not be consistently estimated by any estimation method. First, the system structure has to be known for a consistent estimation to be meaningful. Furthermore, consistency is an asymptotic property, while the effect of the initial conditions on the data vanishes with an increasing data window, ([SödSt89], p.492).

Next, we shall summarize the modifications involved with an output–error method for estimating the channel model. Subsequently, the section will be concluded by an illustration of the effect of unknown initial conditions on the filter design based on a computer simulation. Denote the output prediction by

$$\hat{y}(t) = \frac{B(q^{-1})}{A(q^{-1})} u(t) \quad .$$

If difference equations are implemented with the matlab routine

$$\mathtt{w} = \mathtt{filter}(\mathtt{B}, \mathtt{A}, \mathtt{u}, \theta_{\mathtt{ic}}) \quad , \tag{4.4.5}$$

it should be recalled that the initial conditions in (4.4.5) refer to a transposed direct form II structure given by, see [OppSc89]

$$
\begin{aligned}
x(0) &= \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix}^T = \theta_{ic} \\
x(t) &= \begin{bmatrix} -a_1 & 1 & & 0 \\ -a_2 & 0 & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ -a_n & 0 & \dots & 0 \end{bmatrix} x(t-1) + \begin{bmatrix} b_1 - b_0 a_1 \\ \vdots \\ b_n - b_0 a_n \end{bmatrix} u(t) \qquad (4.4.6) \\
\hat{y}(t) &= \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} x(t-1) + b_0 u(t) \ .
\end{aligned}
$$

Decompose the signal $\hat{y}(t)$ as

$$\hat{y}(t) = \hat{y}_o(t) + \hat{y}_{\text{ic}}(t)$$

where $\hat{y}_o(t)$ depends only on the input signal for $t > 0$ with initial condition equal to zero, and $\hat{y}_{\text{ic}}(t)$ depends only on the initial conditions with no input signal. It is then easily shown that the derivatives of the output prediction $\hat{y}(t)$ are obtained by

$$\frac{\delta}{\delta b_i}\hat{y}(t,\theta_1) = \frac{q^{-i}}{A(q^{-1})}u(t) \ ; \ x(0) = \mathbf{0} \ ; \ i = 0,\dots,nb \qquad (4.4.7)$$

$$\frac{\delta}{\delta a_i}\hat{y}(t,\theta_1) = -\frac{q^{-i}}{A(q^{-1})}\hat{y}(t) \ ; \ x(0) = \theta_{ic} \ ; \ i = 1,\dots,na \qquad (4.4.8)$$

$$\frac{\delta}{\delta c_i}\hat{y}(t,\theta_1) = \frac{1}{A(q^{-1})}\bar{u}(t) \ ; \ i = 1,\dots,n \qquad (4.4.9)$$

where the difference equations (4.4.9) are implemented as in (4.4.6) with

$$
\begin{aligned}
\bar{u}(t) &\equiv 0 \quad , \quad \forall t \\
c_i &= 1 \quad , \quad c_{j \neq i} = 0 \ .
\end{aligned}
$$

Observe that it is not necessary to implement all the filters in (4.4.9) for $i = 1,\dots,n$, because the following relation holds:

$$\frac{\delta}{\delta c_{i+1}}\hat{y}(t) = \frac{\delta}{\delta c_i}\hat{y}(t-1) \ .$$

The effect of unknown initial conditions will be illustrated in the following example for a nominal design of a linear deconvolution filter. It is shown that an explicit estimation of the initial conditions may reveal to be rather ineffective, and that the effect of an unknown initial input different from zero does not seem to significantly modify the performance of a design based on the assumption of a zero input.

Example 4.2 Effect of unknown initial conditions. Experiments were conducted on 10 systems from each of the two sets, $\mathcal{S}_{ni}$ and $\mathcal{S}_{na}$, described in Section 5.4. The channels in the set $\mathcal{S}_{na}$ have poles and zeros closer to the unit circle than those in $\mathcal{S}_{ni}$. The effect of unknown initial conditions will be thus more pronounced for the former systems. In both sets, the channel transfer functions have equal polynomial orders

$$na_T = nb_T = 12 \ ,$$

Set $\mathcal{S}_{ni}$						
ν	ic		u		u_0	
level	ρ	ρ_T	ρ	ρ_T	ρ	ρ_T
20 %	15	26	34	37	26	35
50 %	64	74	84	85	82	86
100 %	90	90	97	97	96	98
0.05	76	81	95	95	91	93

Set $\mathcal{S}_{na}$						
ν	ic		u		u_0	
level	ρ	ρ_T	ρ	ρ_T	ρ	ρ_T
20 %	11	12	21	21	20	22
50 %	51	63	75	71	74	79
100 %	72	80	95	94	95	99
0.05	53	66	77	77	77	81

Table 4.2: Example 4.2: effect of initial conditions, filtering estimator. Left table: nice–systems $\mathcal{S}_{ni}$. Right table: nasty–systems $\mathcal{S}_{na}$. Percentage of filters resulting in a performance degradation less than ν, as compared to the optimal attainable performance for the system for which they are designed. The ν–levels are indicated in the first column. First three rows: percentage degradation of performance. Fourth row: absolute degradation of performance. Each set is composed by 10 systems. Design cases: 100. White noise with SNR = 13dB. DATA = 50.

and output measurements affected by white noise. The signal to noise ratio at the channel output was set to SNR = 13dB. For all the systems and noise configurations, the optimal attainable performance satisfies $V_W \leq 0.01$. Filtering estimators were designed with an indirect method based on a nominal design. The channel model was estimated of order

$$na = nb = 5$$

while the noise was known to be white. An input different from zero was used before the experiment. In the case denoted by "ic", the initial conditions were estimated, and used to estimate the noise variance by computing the residual sequence as

$$\mathtt{v} = \mathtt{y} - \mathtt{filter}(\mathtt{B}, \mathtt{A}, \mathtt{u}, \theta_{\mathtt{ic}}) \quad , \tag{4.4.10}$$

for the estimated channel model polynomials B and A. In the case denoted by "u", the standard two–stage identification algorithm introduced in the previous section was used with initial conditions set as in the Matlab routines described above. In the case denoted by "u_0", the same estimation was repeated after elimination on the output signal of the effect of the past input values before the identification experiment. The latter case corresponds to the situation that will be considered in the simulation studies in Chapter 5. In order to investigate the quality of the channel estimate, filters were also designed by using the true noise variance ρ_T.

For each channel, filters were designed for 10 different input and noise realizations.

The experiment outcome is shown in Table 4.2. The choice of setting the initial conditions as in the Matlab routines results in a considerable improvement of performance as compared to estimating them. The performance statistics obtained in the case "u" and "u_0" are comparable. Hence, the effect of an unknown input different from zero does not seem to significantly influence the attainable designs based on 50 data, for the considered set of systems ■

4.5 Model Validation

The problem of model validation represents a key issue in system identification. Once a model structure has been chosen and the parameter estimate has been obtained, the question is whether the obtained model is "good enough or not good" for the intended purpose. A related problem is the structure selection, that corresponds to deciding which of two model classes should be chosen. Both problems have been extensively studied in the literature. For a comprehensive account, see, for instance, ([Lju87], Chapter 16) and ([SödSt89], Chapter 11).

The problem of model validation is known to be complex. A large amount of data may be required, preferably including data that were not used for estimation. Several model structures are to be compared and evaluated. The common strategy is to first obtain various models, and then to evaluate them against each other by means of statistical tests or a specified model performance. A reasonable choice for the model classes to be considered is to use a nested set of classes, specified by the orders of the parameter vectors

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3 \subset \dots \quad . \tag{4.5.1}$$

The problem is then called *order model selection*. The use of the standard estimation method introduced in Section 4.3 is suitable for forming a chain of model structures as in (4.5.1). First, the best order for the channel model is estimated, then the same is repeated for increasing orders of the ARMA noise model. The use of a prediction error method, where the channel and the noise models are estimated simultaneously, would correspond to the use of a more complicated parameter set, involving the need of investigating more combinations of different polynomial degrees.

Three approaches to model validation can be distinguished.

The first approach consists of *cross–validation*. It can be utilized when data that were not used for estimation are available. In that case, decisions are made on the basis of how the model behaves when applied on the new data, for instance by computing the loss–function used by the identification algorithm.

The second approach is of interest when the model validation has to be performed on the same data that were used for estimation. In that case, models with richer structure will in general result in a better fit, since more degrees of freedom are available due to the larger set of parameters. That property is known as *overfit*, the model will adjust "too much" to the specific realization obtained from the experiment, and may well fit poorly on different data. In order to limit overfit, model validity criteria that penalize the model complexity, or the model order for the chain (4.5.1), have been considered. For prediction error based methods, one of these criteria, the Final Prediction Error criterion (FPE), is available in the standard routines of the System Identification Toolbox of Matlab, [Matlab]. The criterion to be minimized is defined as

$$\mathrm{FPE}(\theta, N_d) = W_{id}(\theta)\frac{1 + p/N_d}{1 - p/N_d} \tag{4.5.2}$$

where $W_{id}(\theta)$ is the loss–function of the identification algorithm computed for the parameters θ, N_d is the number of data, and p is the number of the model parameters.

In the third approach the model structure is maintained fixed, and the interest is instead focused on studying the class of systems for which the model structure is meaningful. For instance, Ljung suggests the use a 4th order ARMA model as a good first attempt for modeling, [Lju93]. The model fitting will then hardly be improved by increasing the model order, except for very resonant systems. If such models provide a bad fit to the experimental data, other aspects should be taken into account, e.g. non linearities or the presence of other input signals.

In the simulation studies of Chapter 5, the three approaches to model validation described above will all be considered. In situations with short data sequences, it will be shown that the third strategy may well provide better results than the use of the FPE criterion. If validation data sequences are long enough, cross–validation is a very effective strategy. It represents the attainable limit in terms of obtaining the best design out of models chosen in different model classes.

Chapter 5

Experiments on Simulated Data

5.1 Introduction

In this chapter we will, by means of computer simulations, evaluate and compare the methods discussed in the previous chapters for the solution of the deconvolution problem. Several experiments are carried out, with three main purposes:

- to evaluate the theoretical analysis on simulated data, generated to cover a wide variety of test situations,
- to possibly draw some general conclusion on differences in performance between methods for model estimation and filter design,
- to obtain a basic point of reference for further experiments and to investigate directions for further research.

Sections 5.2 – 5.4, serve as introduction to the experiments. In Section 5.2, the experiments are summarized, while in Section 5.3, the basic identification algorithms and notations will be described.

Two problems are studied, namely, the design of a filtering estimator ($m = 0$) and that of a smoother with $m = 4$. Emphasis is given to situations with short data sequences, with only 50 samples of input and output data, with signal to noise ratio SNR = 13dB, available for the design. The underlying systems have been chosen of relatively high order (12 poles and 12 zeros) and with various complexity, to investigate the effect of undermodeling, and an indicative statistics of the performance of various design methods on the whole set of systems was noted. The systems that constitute the design scenarios will be described in Section 5.4.

The design of linear deconvolution filters is investigated in Sections 5.5 – 5.7. Section 5.5 deals with the *direct method*, while Section 5.6 deals with the *robust indirect method* introduced in Section 2.5, the cautious Wiener filter. Finally, Section 5.7 deals with the *nominal indirect design* and the *indirect methods* introduced in Section 2.4.

The design of decision feedback equalizers is investigated in Section 5.8, where the use of the indirect methods introduced in Section 3.3, namely the nominal design and the multiple prediction method, will be considered.

The simulation study is concluded by an experiment conducted in the sufficient order case, summarized in Section 5.9.

Our general conclusions from these experiments are briefly summarized below.

Linear Deconvolution Filters

- Optimal filters are not particularly sensitive. Although high order systems and few noisy data have been utilized, the investigated design problem does not seem to be difficult, in general. Perturbations of the optimal filter of the same magnitude as those resulting from adaptation to observed (short and noisy) data sequences preserve a rather good performance.
- The number of data from which the design is carried out, more than the noise level, seems to be the main factor determining the effectiveness of the design. Sets of 300 data constitute sufficient information for an effective design in the considered scenarios.
- The performance statistics attained by the direct adaptation of the filter to the data is difficult to outperform. The main drawback of the direct method is the occurrence of cases with a very poor performance, outliers.
- The robust design is complicated and its use is not motivated by an improved performance as compared to the use of a nominal design.
- Model validation represents a key issue for an effective use of indirect methods. After a reasonable validation, models obtained by system identification are models "good enough" for the design of deconvolution filters, also when the identification experiment is performed with relatively few, and noisy, data, and the underlying system is complex. A nominal indirect design based on well validated models may outperform the direct method. As compared to the use of direct methods, the indirect methods result, in general, in a less sensitive design. They seem to be more effective for the design of smoothers than for the design of filtering estimators.
- If the noise is colored, it is worthwhile to fit an ARMA model to the noise spectrum, even in situations with short data sequences.
- Good model identification, and in particular good model validation, seems to be the crucial factor for a successful use of indirect methods. It is definitely more important than the introduction of more complicated methods in the filter design stage.

Decision Feedback Equalizers

- Optimal designs of DFEs seem to be rather sensitive. The problem of designing DFEs reveals to be more difficult than linear filter design, in general. Perturbations of the optimal filters of the same magnitude as those resulting from indirect design based on estimated models may result in a severely degraded performance, as compared to the optimal attainable one.

- Strategies for the optimization of the modeling stage seem to require large data sets to be meaningful.
- The use of the multiple prediction method improves the performance attained by a nominal design, but not substantially.
- Improvements of the identification algorithms, both for parameter estimation and model validation, seem to be required for a more effective design of IIR DFEs. The use of multiple predictors estimated independently should also be investigated in further studies.

5.2 List of Experiments

The following experiments will be illustrated in the next sections.

Linear Deconvolution Filters

Experiment 5.1: Direct method. Filtering estimator, noise level SNR = 20dB.

Experiment 5.2: Direct method. Filtering estimator, noise level SNR = 13dB.

Experiment 5.3: Direct method. Smoothing, with $m = 4$.

Experiment 5.4: Is the estimated parameter covariance matrix of use for robust design?

Experiment 5.5: Does the cautious filter have to be tuned?

Experiment 5.6: Nominal vs robust indirect design.

Experiment 5.7: Is the two–stage indirect method a sensible approach?

Experiment 5.8: Indirect methods. Filtering estimator.

Experiment 5.9: Indirect methods. Smoothing with $m = 4$.

Experiment 5.10: The spectrum method.

Decision Feedback Equalizers

Experiment 5.11: DFE for filtering. Fixed order models.

Experiment 5.12: DFE for filtering. Cross–validated models.

Sufficient Order Case: Design of a filtering estimator.

5.3 Identification Algorithms and Notations

In this section we will summarize the basic identification algorithms and the notation utilized in the experiments.

Data are generated by a given linear, time–invariant and stable system

$$\begin{aligned} y(t) &= G_T(q^{-1})u(t) + v(t) \\ \mathrm{E}u^2(t) &= 1 \end{aligned} \tag{5.3.1}$$

where the input signal $u(t)$ is a gaussian white noise with zero mean. In the experiments related to DFE design, the input $u(t)$ is a white binary signal, with equally likely symbols $\{-1, 1\}$. The measurement noise $v(t)$ in (5.3.1) is stationary, and is generated as a filtered white noise

$$\begin{aligned} v(t) &= H_T(q^{-1})e(t) \\ \mathrm{E}e^2(t) &= \rho_T \end{aligned} \tag{5.3.2}$$

where $H_T(q^{-1})$ is a stable, linear and minimum phase transfer function, and where the stochastic process $e(t)$ is a gaussian white noise with zero mean, uncorrelated with the input signal $u(t)$. The channel and noise configurations considered in the experiments are described in the next section.

The input and output sequences in (5.3.1) are available for the period

$$t = 1, \ldots, N_d \quad ,$$

and they are both zero before the observation period. That choice has been made to reduce the model error due to unknown initial conditions, see Section 4.4. The situation could represent an identification experiment performed on a system that was previously disconnected.

Models have the general structure

$$\begin{aligned} y(t) &= \frac{B(q^{-1})}{A(q^{-1})}u(t) + \frac{M(q^{-1})}{N(q^{-1})}\hat{e}(t) \\ \rho &= \mathrm{E}\hat{e}^2(t) \end{aligned} \tag{5.3.3}$$

where the polynomial orders of $B(q^{-1})$ and $M(q^{-1})$ are equal to those of $A(q^{-1})$ and $N(q^{-1})$, respectively. The polynomial $M(q^{-1})$ is stable and monic. The model orders will be denoted by

$$\begin{aligned} nG &\triangleq na = nb \\ nH &\triangleq nn = nm \quad . \end{aligned}$$

Two basic identification algorithms are utilized for model estimation. The first algorithm is the standard two–step algorithm introduced in Section 4.3, which is based on estimating the channel model with an output error method and then fitting an ARMA model to the model residual by a prediction error method. This algorithm will be utilized for the design of both linear deconvolution filters and decision feedback equalizers.

The second algorithm is a one–step estimation algorithm, in which the channel and the noise models are estimated in (5.3.3) simultaneously, by minimizing the variance of the one step ahead prediction error of the output signal $y(t)$

$$\tilde{y}(t|t-1) = b_0 u(t) + \hat{e}(t) \quad . \tag{5.3.4}$$

The coefficient b_0 in (5.3.4) is the leading coefficient of the polynomial $B(q^{-1})$ in (5.3.3). The minimization of the variance of $\tilde{y}(t|t-1)$ is performed by the use of a standard prediction error method, where the variance of $\hat{e}(t)$ in (5.3.3) is minimized, after the appropriate modification of the derivative of the loss function with respect to the parameter b_0. This algorithm will be utilized only for the design of decision feedback equalizers. In the case of filtering, it was shown in Section 3.4 that such an estimation method will asymptotically have a minimum of the loss function corresponding to the optimal model for DFE design within a given model class.

The direct method for the design of a linear deconvolution filter is based on an output error method applied to the model equation

$$u(t-m) = \frac{Q(q^{-1})}{R(q^{-1})} y(t) + z(t) \tag{5.3.5}$$

where the polynomials $Q(q^{-1})$ and $R(q^{-1})$ have same polynomial order. The filter order will be denoted by

$$nF \triangleq nq = nr \quad .$$

The algorithms use a loss function that is robustified against the occurrence of outliers in the data. In all the algorithms, difference equations are initialized as in the standard routines available in the System Identification Toolbox for Matlab, [Matlab]. See Section 4.4.

Each experiment is organized into two distinct parts. In the first part the experiment is described. In the second part, under the title "discussion", the outcome of the experiment is summarized and evaluated. The experiment outcomes will be given in tables. Experiments are conducted on a large number of systems, and the outcomes of interest are significant performance differences for the various methods. Nearly all tables report the same type of results, namely the percentage of cases that resulted in a certain maximal performance degradation with respect to the optimal attainable one. This type of performance evaluation is based on the concept of the feasible filter set of ν–level for a given system, introduced in Section 2.3.

Each design method is denoted by

$$\mathcal{D}_{met}$$

for the relevant method "*met*". The specific notation will be introduced the first time a method is encountered.

5.4 Design Scenarios for the Experiments

With the aim of drawing some general conclusions for the design methods, experiments are conducted on data generated by a large number of channel and noise configurations, that represent rather general sets of design scenarios. We will describe those scenarios in this section.

In Chapter 2, it was argued that systems with resonance peaks and deep notches in the frequency function may lead to a sensitive optimal design of a linear deconvolution filter. In order to distinguish less sensitive (i.e. easier) design cases from more difficult ones, two different sets of channel and noise configurations S were randomly generated. The two sets represent design scenarios where the channels have similar characteristics[1]. To investigate the effect of undermodeling on the attainable design, systems to which correspond an optimal deconvolution filter of high order were considered.

In both sets, the channel transfer function $G(q^{-1})$ has an ARMA structure with a total of 25 parameters and equal polynomial degrees

$$nG_T \triangleq nb_T = na_T = 12 \quad .$$

The noise is colored, and described by ARMA innovation models with equal polynomial degrees

$$nH_T \triangleq nm_T = nn_T$$

randomly selected so that

$$1 \ \leq \ nH_T \ \leq \ 10 \quad .$$

The optimal linear deconvolution filters for all the design cases in the two sets will have transfer functions with equal polynomial degrees, denoted nF_W, that satisfy

$$13 \ \leq \ nF_W \ \leq \ 22 \quad ,$$

except for cases with exact zero–pole cancellations. All the channels in both sets have been selected so that, when the output is measured in the presence of an additive white noise with signal to noise ratio SNR = 13dB, the performance of the optimal filtering estimator satisfies

$$V_W \ \leq \ 0.2 \quad . \tag{5.4.1}$$

Recall, from the previous section, that the input is white and with unit variance. The trivial estimator $\hat{u} \equiv 0$ will thus result in a performance equal to one. The reason for the choice above is that it seems unrealistic to assume that linear deconvolution filter or equalizer design would be considered in situations with a lower attainable performance. The noise spectra have been chosen so that the attainable optimal performance under the same conditions as above satisfies

$$V_W \ \leq \ 0.1 \quad .$$

In one set, denoted $\mathcal{S}_{ni}$, where the subscript "*ni*" stands for "*nice*", the zeros and the poles of each channel $G(q^{-1})$ are selected randomly within the circle of radius

$$\rho_C = 0.8 \quad .$$

In the other set, $\mathcal{S}_{na}$, where the subscript "*na*" stands for "*nasty*", either some zero or pole are randomly placed with magnitude ρ_D such as[2]

$$0.87 \ \leq \ \rho_D \ \leq 0.93$$

[1]The sets have already been utilized in Example 2.1 in Section 2.3, and in Examples 4.1 and 4.2 in Chapter 4.

[2]For a larger magnitude, it was very difficult to find channels for which (5.4.1) held. The same applies for nonminimum phase channels. Nonminimum phase channels are considered in the smoothing case.

Figure 5.1: Magnitude Bode plot of typical channel transfer functions and the corresponding optimal filtering estimators in the set $\mathcal{S}_{ni}$, left figures, and $\mathcal{S}_{na}$, right figures. The noise is colored, with $\mathrm{SNR} = 13\mathrm{dB}$.

while other zeros and poles are within the circle of radius

$$\rho_C = 0.85 \quad .$$

In both sets, all channels are thus minimum phase. The channel transfer functions have been normalized to have an unitary H_2–norm. The "nasty" set $\mathcal{S}_{na}$ is expected to represent more difficult design cases than those in the "nice" set $\mathcal{S}_{ni}$, because of zeros and poles close to the unit circle. See Example 2.1. The magnitude Bode plot of typical channel transfer functions in both sets are shown in Figure 5.1, together with the corresponding optimal filtering estimator. It is interesting to note that the optimal filter almost inverts the channel dynamics, as was assumed to be the case in Section 2.4.

The performance difference between filters and smoothers is rather small for the two sets considered above. Channels for which deconvolution is improved by smoothing will have small leading impulse response coefficients as compared to the following ones. In order to select channels suitable for investigating the effect of smoothing, channels selected from the two sets described above had the first impulse response coefficient reduced to a small percentage of the second one. Note that in this way, the positions of zeros are modified, and nonminimum phase channels will frequently be obtained. The noise spectra were

Figure 5.2: Magnitude Bode plot of typical channel transfer functions and the corresponding optimal smoother with $m = 4$ in the set $\mathcal{S}_{ni}$, left figures, and $\mathcal{S}_{na}$, right figures, for smoothing. The noise is colored, with $\mathrm{SNR} = 13\mathrm{dB}$.

then selected so that the performance of the optimal smoother with $m = 4$ satisfied

$$V_W \leq 0.1 \quad .$$

With the strategy above, two sets of design cases, $\mathcal{S}_{ni}$ and $\mathcal{S}_{na}$ for the evaluation of smoothing, were thus generated. It is worth mentioning that the nasty systems really turned out to create difficult design cases.

The magnitude Bode plot of typical channel transfer functions in both sets are shown in Figure 5.2, together with the corresponding optimal smoother with $m = 4$.

The sets $\mathcal{S}_{ni}$ and $\mathcal{S}_{na}$ for filtering and smoothing constitute the design scenarios also for the evaluation of decision feedback equalizers.

5.5 The Direct Method

In this section we shall study the use of a direct method, introduced in Section 1.4, for the design of a linear deconvolution filter. If the number of data available for the design is large enough, the use of a direct method will, in principle, provide the optimal attainable design with a given filter structure. The direct method is thus a good candidate for the design of a deconvolution filter based on observed data. Furthermore, in Section 2.3 it was shown that optimal filters of high order can mostly be approximated by low order filters with only a small loss of performance. In Section 4.2, it was however indicated that the direct method may suffer from a high sensitivity to the specific noise realization during the identification experiment.

We will conduct three experiments.

In Experiment 5.1, the case of filtering ($m = 0$) with a low level of measurement noise will be investigated. As could be expected, it will be shown that the available number of data does strongly limit the number of estimated parameters, in the most effective design. This restriction sets an important limit on the attainable performance in situations with few training data. Cases with very poor performance, *outliers*, are observed in situations with short data sequences.

In Experiment 5.2, the Experiment 5.1 is repeated with a higher noise level. It will be shown that, while the number of parameters that can be estimated is not reduced by the increased noise level, the occurrence of outliers becomes more frequent. If the number of data is large, the increased noise level does not appreciably alter the distribution of the relative performance deteriorations.

In Experiment 5.3, the case of smoothing ($m = 4$) will be investigated. The conclusions from the previous experiments are shown to be valid also for smoothers.

The number of data from which the design is carried out, more than the noise level, reveals to be the major limiting factor for the use of a direct method. The use of short data sequences influences three interrelated aspects of the design:

- The design results in a lower performance for any given order of the filter.
- The number of parameters that can be estimated in a successful design is reduced.
- The percentage of outliers increases.

The result of these experiments motivate the study of alternatives to the direct method, in particular for situations with short data sequences. The use of indirect methods, presented in Chapter 2, will be investigated in the following sections.

Experiment 5.1 Filtering estimator, low noise.

The filter is designed with the direct method. See Section 5.3.

Experiments were conducted on 40 systems from each of the two sets, $\mathcal{S}_{ni}$ and $\mathcal{S}_{na}$, introduced in Section 5.4. Colored noises were used. The signal to noise ratio at the channel output was set to SNR = 20dB. For all the design cases, the optimal attainable performance satisfies

$$V_W \leq 0.025 \quad .$$

Set $\mathcal{S}_{ni}$

DATA = 50						
ν	filter order					
level	3	4	5	6	7	8
20 %	12	16	19	11	6	5
50 %	45	63	55	42	26	20
100 %	63	83	77	66	52	40
0.05	96	94	88	80	77	61
out	2	2	5	9	13	19

DATA = 300						
ν	filter order					
level	3	4	5	6	7	8
20 %	36	68	89	94	89	90
50 %	55	91	100	99	96	97
100 %	76	95	100	99	99	97
0.05	100	100	100	100	99	99
out	0	0	0	0	0	1

Set $\mathcal{S}_{na}$

DATA = 50						
ν	filter order					
level	3	4	5	6	7	8
20 %	1	3	5	6	5	2
50 %	3	18	31	33	26	19
100 %	6	32	55	63	53	47
0.05	39	74	91	85	78	72
out	2	3	5	6	12	12

DATA = 300						
ν	filter order					
level	3	4	5	6	7	8
20 %	2	25	50	74	87	89
50 %	5	34	70	89	95	99
100 %	12	46	88	94	96	99
0.05	50	91	100	99	99	100
out	0	0	0	1	1	0

Table 5.1: Experiment 5.1: direct method, filtering estimator. Left table: nice–systems $\mathcal{S}_{ni}$. Right table: nasty–systems $\mathcal{S}_{na}$. Percentage of filters resulting in a performance degradation less than ν, as compared to the optimal attainable performance for the system for which they are designed. The ν–levels are indicated in the first column. First three rows: percentage degradation of performance. Fourth row: absolute degradation of performance. Last row: percentage of outliers. Number of design cases: 200. SNR = 20dB.

For every system configuration, filters of different orders were estimated 5 times with different input and noise realizations. In total, 200 design cases for each filter order were thus considered. The designs were carried out for two data lengths, DATA = 50 and DATA = 300, respectively.

The experiment outcome is shown in Table 5.1. In the fifth row of the table, corresponding to the label "out", the percentage of outliers is reported, where an outlier is defined as a filter that results in a performance

$$V(S_T, \mathcal{D}) \geq 1 \ .$$

A performance equal to 1 is attained by the trivial estimate $\hat{u}(t) = 0$.

Discussion. In Table 5.1, the effect of the duration of the identification experiment on the performance is apparent for systems in both sets. A larger data set improves the design in three interrelated ways:

- The performance improves for all the considered orders for the designed filters.
- Filters of higher order can be successfully estimated. Moreover, they result in a considerably better performance than lower order filters.
- The occurrence of outliers is almost eliminated in the experiments using 300 data.

ν level	filter order 3	4	5	6	7	8
DATA = 50						
20 %	28	25	18	8	3	1
50 %	71	63	53	38	19	17
100 %	86	82	76	59	42	35
0.05	88	79	73	59	39	31
out	3	6	11	16	22	26
DATA = 300						
20 %	73	93	95	94	88	89
50 %	89	99	99	97	95	96
100 %	95	100	100	98	98	97
0.05	99	100	100	98	98	98
out	0	0	0	1	2	1
Set $\mathcal{S}_{ni}$						

ν level	filter order 3	4	5	6	7	8
DATA = 50						
20 %	2	8	11	3	3	2
50 %	12	32	41	36	24	16
100 %	35	66	67	61	48	37
0.05	30	53	60	53	40	28
out	3	5	10	14	18	29
DATA = 300						
20 %	11	38	76	89	90	89
50 %	28	74	96	99	98	98
100 %	60	97	99	99	99	98
0.05	48	89	98	99	99	98
out	0	0	0	1	2	1
Set $\mathcal{S}_{na}$						

Table 5.2: Experiment 5.2: direct method, filtering estimator. Left table: nice–systems $\mathcal{S}_{ni}$. Right table: nasty–systems $\mathcal{S}_{na}$. Percentage of filters resulting in a performance degradation less than ν, as compared to the optimal attainable performance for the system for which they are designed. The ν–levels are indicated in the first column. First three rows: percentage degradation of performance. Fourth row: absolute degradation of performance. Last row: percentage of outliers. Number of design cases: 200. SNR = 13dB.

It is, however, worth noticing that the filter performance in maintained at acceptable levels also with few data. See the fourth rows of Table 5.1. The design of low order filters, 3rd order for systems in $\mathcal{S}_{ni}$ and 5th order for systems in $\mathcal{S}_{na}$, resulted in a degradation of performance limited by

$$V(S_T, F_{dir}) \;\leq\; V_W + 0.05$$

for more than 90% of the designs based on 50 data.

It is interesting to note that the "nasty" systems in the set $\mathcal{S}_{na}$ do represent more difficult design cases than those in the "nice" set $\mathcal{S}_{ni}$. This aspect was already noted in Example 2.1, Section 2.3, where the problem was to approximate the known optimal filter with a low order filter. The values reported in Table 5.1 can also be compared to those in Table 2.1 in Example 2.1. Note, however, that the latter case refers to a signal to noise ratio SNR = 13dB. It is rather surprising that with "only" 300 data observed from an unknown system, the design problem can be solved as effectively as indicated by Table 5.1.

In the next experiment, we will investigate the influence of the noise level on the performance ∎

Experiment 5.2 Filtering estimator, high noise.

In this experiment we repeat the Experiment 5.1 with SNR = 13dB. The channel and noise configurations are the same as in the Experiment 5.1. With the increased noise level, the optimal attainable performance satisfies

$$V_W \leq 0.1$$

in each case. The experiment outcome is shown in Table 5.2.

Discussion. As compared to the results of the Experiment 5.1, the designs result in a lower percentage degradation of performance, for filters of order $3 - 5$. Note, however, that the attainable optimal performance is, on average, degraded by a factor of 4 as compared to the previous case. In terms of absolute values of degradation, the performance deterioration is worse than in the Experiment 5.1 when the data are few, while the levels are similar for DATA = 300. In the latter case, the increased noise level does not increase the performance degradation. Compare to Table 5.1.

The resulting performance degradation seems to be mainly determined by the available number of data. In Table 5.2, observe that with DATA = 50, both poor performances and a high number of outliers are reported, especially for the set $\mathcal{S}_{na}$.

Table 5.2 should be compared to Table 2.1 in Example 2.1. The same design cases are considered[3], and the difference is only in the information available for the design. In Example 2.1 the system configurations were known, and the problem was to obtain a low order approximation of a given optimal filter. In Table 5.2, the design of the low order filters is instead based only on observed data from an unknown system. As was already pointed out in the Experiment 5.1, it is surprising to report that the observation of 300 data is, for this case, sufficient information on the system. Deconvolution filter designed on the basis of such information result in a performance comparable to what could have been attained if the channel and noise spectrum were exactly known ■

Experiment 5.3 Smoothing, with $m = 4$.

The optimal structure for a smoother is given by, see Section 2.2,

$$F(q^{-1}) = Q_1(q^{-1})F_1(q^{-1}) \tag{5.5.1}$$

where $Q_1(q^{-1})$ is an FIR filter of mth order, where m is the smoothing lag, and $F_1(q^{-1})$ is the (IIR) whitening filter.

Experiments were conducted on 40 systems from each of the two modified sets $\mathcal{S}_{ni}$ and $\mathcal{S}_{na}$ for smoothing, introduced in Section 5.4. Colored noises were used. The signal to noise ratio at the channel output was set to SNR = 13dB. All the systems configurations have been selected so that the optimal attainable performance satisfies

$$V_W \leq 0.1 \quad .$$

[3]With the exception that, in Table 2.1, a larger number of systems was considered in each set.

DATA = 50						
ν	filter order					
level	4+3	3	4	5	6	7
20 %	9	6	15	15	6	5
50 %	52	32	55	56	35	27
100 %	75	72	83	80	63	48
0.05	66	57	78	76	59	41
out	10	1	3	6	13	18
DATA = 300						
ν	filter order					
level	4+3	3	4	5	6	7
20 %	96	23	69	98	94	93
50 %	99	45	88	99	98	97
100 %	99	85	100	99	98	98
0.05	99	75	100	99	98	98
out	0	0	0	1	1	1
Set $\mathcal{S}_{ni}$						

DATA = 50						
ν	filter order					
level	4+3	3	4	5	6	7
20 %	7	7	10	11	9	4
50 %	39	25	35	43	39	21
100 %	72	47	64	69	66	48
0.05	60	43	52	58	59	39
out	7	1	2	5	11	16
DATA = 300						
ν	filter order					
level	4+3	3	4	5	6	7
20 %	73	22	42	75	86	94
50 %	99	39	64	85	95	99
100 %	100	60	82	97	99	99
0.05	99	53	77	88	97	99
out	0	0	0	1	0	1
Set $\mathcal{S}_{na}$						

Table 5.3: Experiment 5.3: direct method, smoothing ($m = 4$). Column 2: smoother with structure (5.5.1). Column 3–7: smoothers with polynomial $Q_1(q^{-1})$ in (5.5.1) set to 1. Left table: nice–systems $\mathcal{S}_{ni}$. Right table: nasty–systems $\mathcal{S}_{na}$. Percentage of filters resulting in a performance degradation less than ν, as compared to the optimal attainable performance for the system for which they are designed. The ν–levels are indicated in the first column. First three rows: percentage degradation of performance. Fourth row: absolute degradation of performance. Last row: percentage of outliers. Number of design cases: 200. SNR = 13dB.

The design of filters with the structure (5.5.1), with $m = 4$ and different orders for the filter $F_1(q^{-1})$, were carried out for two data lengths, DATA = 50 and DATA = 300, respectively. For every system configuration, the filters were estimated 5 times, using different input and noise realizations. In total, 200 design cases were thus considered. Since the previous experiments demonstrated that the number of parameters that can be estimated is limited, filters where the polynomial $Q_1(q^{-1})$ in (5.5.1) was set to 1 where also designed. For a given order of the filter $F_1(q^{-1})$, the latter filters will thus have less parameters than those with the structure (5.5.1).

The experiment outcome is shown in Table 5.3, were the performance of the filters with $Q_1 \equiv 1$ is reported for several orders of the filter F_1, while the performance of the filters with the structure (5.5.1) is reported only for a filter F_1 of 3rd order. The latter is shown in the column with the label $4 + 3$. See the Experiment 5.1 for a further explanation of the tables.

Discussion. The indications given by the Experiments 5.1 and 5.2 are confirmed. For a given order of the filter $F_1(q^{-1})$ in (5.5.1), more parameters are estimated in the case of smoothing as compared to the pure filtering case ($m = 0$). When data are few, the number of parameters that can be successfully estimated is limited, and the performance is thus expected to be poor. That was reflected both in terms of performance degradation and outliers. In Table 5.3, the performance of filters with the structure (5.5.1) is reported only for a filter F_1 of 3rd order, which resulted in the best performance among the order $nF_1 = 3, \ldots, 8$. No significant differences in the patterns of performance degradation as compared to the filtering case are observed. Note, however, that the number of outliers is reduced by a factor of 2. Compare to Table 5.2. As was argued in Section 4.2, the use of smoothing seems to decrease the sensitivity of the estimation to the specific noise realizations which affect the measurements. As in the filtering case, 300 data are adequate for obtaining effective designs ■

5.6 Cautious Filters and Robustification

In the previous section, the direct method was shown to result in degraded performances when the number of data available for the design is limited. In this section we shall study the use of the cautious Wiener filter introduced in Section 2.5 as an alternative to the direct method. The possible benefit of robust designs as compared to nominal designs will also be investigated. The reader may want to review Section 2.5 before proceeding.

When using the cautious Wiener filter, two aspects need to be investigated.

The first aspect is whether the second order model description (SOMD) obtained from an estimate of the parameter covariance matrix is meaningful when undermodeling is present. That will be addressed in Experiment 5.4. It will be shown that meaningful results can indeed be obtained by using the simple "standard" estimate (2.5.24), also for short data sequences.

The second aspect is whether scalar tuning parameters γ_G and γ_H in the extended noise model (2.5.25) should be utilized when models are obtained by system identification. This question arises due to the distinction between design Case 1 and design Case 2, and will be addressed in Experiment 5.5. It will be shown that the tuning of these parameters is, in fact, required to optimize the performance.

The design of the cautious filter is more complicated than nominal designs. Hence, it is of interest to investigate whether the increased complexity is motivated by a corresponding improvement of the deconvolution performance. That will be addressed in Experiment 5.6. It will be shown that nominal designs reveal performances comparable to those attained by cautious filters. That result will be interpreted as due to the following reason:

> Models obtained by system identification, after a reasonable validation, are "good enough" for the design of Wiener filters, also when the identification experiment is performed with relatively few, and noisy, data. A reason for this is that optimal filters are in general not particularly sensitive. Perturbations of the optimal filter of the same magnitude as those resulting from indirect designs based on system identification still result in a rather good performance.

Experiment 5.4 Is the estimated parameter covariance matrix of use for robust design?

In this experiment we investigate whether the SOMD obtained from an estimate of the parameter variance is meaningful when undermodeling is present. We consider the simple estimate obtained by the "standard" estimator (2.5.24). It should be recalled that in the case of undermodeling such an estimate is a biased (and poor) estimate of the parameter variance, [HjaPhD93]. The cautious Wiener filter is designed as follows.

Design of the Cautious Wiener Filter

Step 1 A model S is obtained by the standard estimation method described in Section 5.3.

Step 2 An estimate of the parameter variance is available from Step 1. The estimate is obtained by the use of the "standard" variance estimator (2.5.24) as follows, see [Lju87], [SödSt89]. The channel model is estimated in the model equation

$$y(t) = \frac{B(q^{-1})}{A(q^{-1})}u(t) + v(t) \; ; \; \theta_G = \begin{bmatrix} b_0 & \dots & b_{nG} & a_1 & \dots & a_{nG} \end{bmatrix}^T \quad . \tag{5.6.1}$$

The output prediction based on the model (5.6.1) is formed as

$$\hat{y}(t) = \frac{B(q^{-1})}{A(q^{-1})}u(t) \quad . \tag{5.6.2}$$

Denote the derivative of the output prediction (5.6.2) with respect to the parameter θ_i in (5.6.1) by

$$\psi_{\theta_i}(t, \theta) = \frac{\delta}{\delta \theta_i}\hat{y}(t) \quad .$$

The gradient vector of the output prediction (5.6.2) is then defined as

$$\psi(t, \theta_G) = \begin{bmatrix} \psi_{b_0}(t, \theta_G) & \dots & \psi_{a_{nG}}(t, \theta_G) \end{bmatrix}^T \quad . \tag{5.6.3}$$

From (5.6.3), the estimate of the variance of the estimated parameters $\hat{\theta}_G$ in the model (5.6.1) is obtained by

$$\hat{\mathbf{P}}_G = \frac{1}{N_d}\rho_v \left[\frac{1}{N_d} \sum_{t=1}^{N_d} \psi(t, \hat{\theta}_G)\psi^T(t, \hat{\theta}_G) \right]^{-1} \tag{5.6.4}$$

where N_d is the number of data and ρ_v is the estimated variance of the model residual $v(t)$ in (5.6.1). The estimate $\hat{\mathbf{P}}_H$ of the variance of the parameters in the noise model is obtained similarly as above from the ARMA model

$$v(t) = \frac{M(q^{-1})}{N(q^{-1})}\hat{e}(t) \; , \; \theta_H = \begin{bmatrix} m_1 & \dots & m_{nH} & n_1 & \dots & n_{nH} \end{bmatrix}^T \tag{5.6.5}$$

and the one–step prediction

$$\hat{v}(t|t-1) = \left[1 - \frac{N(q^{-1})}{M(q^{-1})} \right] v(t) \quad . \tag{5.6.6}$$

We will refer to the variance estimates $\hat{\mathbf{P}}_G$ and $\hat{\mathbf{P}}_H$ as the *pem–variance* $\hat{\mathbf{P}}_{pem}$.

Set $\mathcal{S}_{ni}$		
ν	covariance matrix	
level	$\hat{\mathbf{P}}_{pem}$	$\mathbf{P}_T$
20 %	20	10
50 %	54	31
100 %	80	59
0.05	81	58

Set $\mathcal{S}_{na}$		
ν	covariance matrix	
level	$\hat{\mathbf{P}}_{pem}$	$\mathbf{P}_T$
20 %	13	8
50 %	59	45
100 %	80	69
0.05	78	60

Table 5.4: Experiment 5.4. Left table: nice–systems $\mathcal{S}_{ni}$. Right table: nasty–systems $\mathcal{S}_{na}$. Percentage of filters resulting in a performance degradation less than ν, as compared to the optimal attainable performance for the system for which they are designed. The ν–levels are indicated in the first column. First three rows: percentage degradation of performance. Last row: absolute degradation of performance. Each set is composed by 10 systems. SNR = 13dB. DATA = 50.

Step 3 The model estimated at Step 1 and the pem–variance $\hat{\mathbf{P}}_{pem}$ obtained as in Step 2, are converted into a SOMD by the use of the algorithm presented in [Öhrn95]. The SOMD is given by the nominal model

$$S_0 = \{G_0, H_0, \rho_0\} \tag{5.6.7}$$

in (2.5.20) and the transfer functions

$$\frac{B_{11}(q^{-1}, q)}{A_1(q^{-1})A_{1_*}(q)}\ , \quad \frac{M_{11}(q^{-1}, q)}{N_1(q^{-1})N_{1_*}(q)} \tag{5.6.8}$$

in (2.5.5) and (2.5.9). The algorithm is based on series expansions of the model transfer functions around the nominal value of the parameters. A series expansion of first order was used[4].

Step 4 The cautious Wiener filter is then obtained from equations (2.5.14) and (2.5.17)–(2.5.19) solved with the polynomials in (5.6.7) and (5.6.8). The design will be represented by

$$\mathcal{D}_c$$

□

In order to investigate the use of the pem–variance $\hat{\mathbf{P}}_{pem}$, it was compared to the use of the "real" covariance matrices $\mathbf{P}_G$ and $\mathbf{P}_H$ of the estimated parameters. The latter were obtained by repeating the identification experiment at Step 1 with different input and noise realizations, and then computing the sample estimate of the parameter variance. We will refer to the sample estimate as the *true–variance* $\mathbf{P}_T$.

Experiments were conducted on systems selected from each of the two sets, $\mathcal{S}_{ni}$ and $\mathcal{S}_{na}$, introduced in Section 5.4. Colored noises were used. The signal to noise ratio at the

[4]It should be mentioned that the algorithm is linear with respect to multiplication of the covariance matrix by a scalar factor. For example, if B_{11}, A_1 are the polynomials obtained with a covariance matrix $\hat{\mathbf{P}}_G$, then γB_{11} and A_1 will be obtained from a covariance matrix $\gamma\hat{\mathbf{P}}_G$.

channel output was set to SNR = 13dB. For all the system configurations, the optimal attainable performance satisfies

$$V_W \leq 0.1 \ .$$

For each identification experiment, two cautious filters for filtering ($m = 0$) were designed, one with the pem–variance $\hat{\mathbf{P}}_{pem}$ and one with the true–variance $\mathbf{P}_T$, and their performance were compared. A set of 20 input and noise realizations with 50 data was considered, and the experiment was repeated for 10 system configurations. The model orders were set to

$$nG = 5 \ ; \ nH = 3$$

which was found appropriate for the systems under study. The experiment outcome is shown in Table 5.4.

Discussion. The main result of the experiment is that the use of the pem–variance $\hat{\mathbf{P}}_{pem}$ can very well provide meaningful results also in the case of undermodeling. Actually, the use of the pem–variance $\hat{\mathbf{P}}_{pem}$, which is a poor estimate of the parameter variance in the case of undermodeling, seems to provide *better* designs than the use of the real–variance $\mathbf{P}_T$. That result should not be misinterpreted. As will be shown in the Experiment 5.5, the cautious filter needs to be adjusted by the use of tuning parameters γ_G and γ_H in (2.5.25). It will turn out, that when the design is carried out with the real–variance $\mathbf{P}_T$, small values for the tuning parameters would give a better performance. Roughly speaking, the value

$$\gamma_G = 1 \ ; \ \gamma_H = 1$$

used for designing the cautious filter, is a good tuning for the the pem–variance $\hat{\mathbf{P}}_{pem}$, but a bad tuning when using the the real–variance $\mathbf{P}_T$ ■

Experiment 5.5 Does the cautious filter have to be tuned?

This experiment investigates whether the adjustment of the filter by the use of tuning parameters γ_G and γ_H in (2.5.25) should be utilized when models are obtained by system identification. The question arises due to the distinction between design Case 1 and design Case 2.

The robust filter is designed as in the procedure described in the Experiment 5.4, with a modification of Step 4. The extended noise model, instead of being obtained from the expression (2.5.17), is now obtained by the expression (2.5.25)

$$\bar{\rho}\frac{\bar{M}\bar{M}_*}{(\tilde{N}A_1)(\tilde{N}_*A_{1_*})} = \rho_0\frac{M_0M_{0_*}}{N_0N_{0_*}} + \gamma_H\frac{M_{11}}{N_1N_{1_*}} + \gamma_G\frac{B_{11}}{A_1A_{1_*}} \tag{5.6.9}$$

where the scalar tuning parameters γ_G and γ_H are adjusted until a satisfactory performance is attained. The tuning of the cautious filter will be represented by

$$\mathcal{D}_{tc} \ .$$

Observe that the value $\gamma_G = 1$, $\gamma_H = 1$ for the tuning parameters in (5.6.9) will result in the cautious filter described in the Experiment 5.4, while the value $\gamma_G = 0$, $\gamma_H = 0$ will result in the nominal design[5].

[5]If there is no uncertainty, corresponding to $\gamma_G = \gamma_H = 0$, the returned SOMD will have a nominal model equal to the estimated model, and variance terms in (5.6.9) equal to zero.

covariance $\hat{\mathbf{P}}_{pem}$						
	γ_H					
γ_G	0	0.3	0.6	1	1.5	2
0	16	1	1	1	2	10
0.3	4	1	1	1	1	3
0.6	3	0	0	1	1	3
1	3	0	1	1	2	4
1.5	2	1	1	0	1	4
2	3	1	1	3	2	23

covariance $\mathbf{P}_T$						
	γ_H					
γ_G	0	0.3	0.6	1	1.5	2
0	23	7	5	5	4	9
0.3	10	2	2	2	1	4
0.6	5	1	1	1	1	2
1	3	1	1	1	1	2
1.5	2	1	1	0	0	1
2	2	0	0	1	0	1

Table 5.5: Experiment 5.2. Case–by–case performance. The percentage of cases over the whole experiment in which a certain combination of values for the tuning parameters resulted in the best performance, among the choices indicated in the table. Experiment with 50 systems in $\mathcal{S}_{ni}$. Number of design cases: 1000. SNR = 13dB. DATA = 50.

covariance $\hat{\mathbf{P}}_{pem}$						
	γ_H					
γ_G	0	0.3	0.6	1	1.5	2
0	0	0	0	0	0	2
0.3	0	1	0	2	1	0
0.6	0	0	2	2	3	3
1	0	1	1	1	4	2
1.5	1	1	2	0	1	6
2	2	0	2	1	1	8

covariance $\mathbf{P}_T$						
	γ_H					
γ_G	0	0.3	0.6	1	1.5	2
0	4	7	6	1	1	0
0.3	15	3	1	2	1	0
0.6	4	0	0	0	1	1
1	0	0	1	0	0	0
1.5	0	2	0	0	0	0
2	0	0	0	0	0	0

Table 5.6: Experiment 5.2. Average performance. The number of times a certain combination of values for the tuning parameters resulted, for each system, in the best *average* performance, among the choices indicated in the table. Experiment with 50 systems in $\mathcal{S}_{ni}$. Averages are taken over 20 designs cases for each system. SNR = 13dB. DATA = 50.

In Section 2.5, we pointed out that the need of the tuning remains to be proved. With that aim, the Experiment 5.4 was repeated for several choices of the tuning parameters γ_G and γ_H in (5.6.9). In order to obtain a better statistics, designs were carried out for 50 systems. Only systems in the set $\mathcal{S}_{ni}$ were investigated.

Two different aspects were considered. First the case–by–case performance, and for a given design case, the combination that resulted in the best performance among a set of choices for the tuning parameters was noted. The experiment outcome is shown in Table 5.5, where the percentage of cases in which a certain combination resulted in the best performance among the choices indicated in the table is reported. The second aspect was the average performance, that is the criterion for which the robust filter is designed. For each system configuration, the combination of tuning parameters that resulted in the best average performance over 20 data realizations was then noted. Out of the 50 system configurations, the number of cases in which a certain combination resulted in the best average performance is reported in Table 5.6.

Discussion. The tuning of the filter given in (5.6.9) reveals to be required for obtaining the best robust design in terms of both absolute and average performances. The result can be interpreted as a partial confirmation of the distinction between the two design cases, Case 1 and Case 2. When the "real" parameter variance $\mathbf{P}_T$ is used for designing the robust filter, the choice

$$\gamma_G = 1 \ , \ \gamma_H = 1 \quad ,$$

which solves the design Case 1, almost never results in a best design, and other values for the tuning parameters should be utilized. When the estimated parameter variance $\hat{\mathbf{P}}_{pem}$ is used instead, the quality of the estimate clearly determines the design, and there are not clear indications a–priori for an adequate value of the tuning parameters. The choice of setting them to one turns out to be rather arbitrary. The combination of values for the tuning parameters that resulted in the best average performance over the whole experiment was

$$\gamma_G = 1 \ , \ \gamma_H = 1.5 \quad ,$$

when the design is based on the pem–variance $\hat{\mathbf{P}}_{pem}$, and

$$\gamma_G = 1.5 \ , \ \gamma_H = 0.3 \quad ,$$

when the design is based on the real–variance $\mathbf{P}_T$. It should however be pointed out that it has not been investigated whether the tuning result in a considerably improved performance as compared to the use of the cautious filter. An indication of the possible improvement will be given in the next experiment, where different designs will be compared in terms of the attained performances.

It is interesting to note that the intuition based on the "doubling of the parameter variance" for the description of the model uncertainty, as suggested by the results in [GuLju94], does not seem to apply to the cautious filter. If it applied, the optimal tuning parameters, when the real–variance $\mathbf{P}_T$ is used for the design, would be distributed around the values

$$\gamma_G = 2 \ , \ \gamma_H = 2 \quad .$$

It is not clear whether some rule for a good tuning could be drawn in general. The tuning will depend on the model orders, the design cases and the estimate of the parameter variance. Note, also, that by averaging the performance over different sets of performances, different optimal values for the tuning parameters are obtained. Compare Table 5.6 to the optimal values relative to the whole experiment ■

Experiment 5.6 Nominal vs Robust Design.

In this experiment we investigate whether the use of the robust method results in a performance improvement that motivates the increased complexity as compared to the use of a simpler nominal design.

The following designs will be considered.

1. **Cautious filter**, $(\mathcal{D}_c)$. The cautious filter is designed with the use of the pem–variance $\hat{\mathbf{P}}_{pem}$.

2. **Tuned cautious filter**, $(\mathcal{D}_{tc})$. The filter is designed as in (5.6.9) with the use of the pem–variance $\hat{\mathbf{P}}_{pem}$, and with tuning parameters selected as

$$\gamma_G = 1.5 \ , \ \gamma_H = 0.6 \quad ,$$

which turned out to be the choice, in an experiment as Experiment 5.5 with a different number of systems, that attained the best average performance over the whole experiment.

3. **Nominal filter**, $(\mathcal{D}_n)$. The filter is designed as the Wiener filter for the estimated model. The design is represented by $\mathcal{D}_n$.

4. **White noise filter**, $(\mathcal{D}_{nw})$. The filter is designed as the Wiener filter for the estimated channel model (5.6.1) assuming the noise was white. The design is represented by $\mathcal{D}_{nw}$.

5. **True noise filter**, $(\mathcal{D}_{nT})$. In [StAh93], it was shown that a major feature of the cautious filter was to reduce the occurrence of situations with a very large performance degradation as compared to the use of the nominal design. Occurrence of outliers was observed with the direct method. See the previous section. If the two nominal methods above do *not* result in outlier cases, a possible reason could be that the design is "self–robustified" by estimating the noise model. Hence, it can be of interest to investigate what would happen if the true noise model were used together with the estimated channel model. Would that result in a sensitive design? This design is represented by $\mathcal{D}_{nT}$ □

Experiments were conducted on the same design scenario as in Experiment 5.2. From each of the two sets introduced in Section 5.4, 40 systems where considered with colored noises. The signal to noise ratio at the channel output was SNR = 13dB. For all the system configurations, the optimal attainable performance satisfies

$$V_W \ \leq \ 0.1 \quad .$$

For every system configuration, 5 models with orders

$$nG = 5 \ , \ nH = 3$$

were estimated from different input and noise realizations with 50 data, and an estimator for pure filtering ($m = 0$) was designed with the methods described above. In total, 200 design cases were thus considered.

The outcome of the experiment is summarized in Table 5.7. In the fifth row, corresponding to the label "out", the percentage of outliers is reported, where an outlier is defined as a filter that results in a performance

$$V(S_T, \mathcal{D}) \ \geq \ 1 \quad .$$

The results in Table 5.7 should be compared to those for the direct method in Table 5.2.

Discussion. The nominal design results in performances comparable to those attained by the robust designs. The tuning of the robust filter does not result in a significant

Set $\mathcal{S}_{ni}$					
ν	design method				
level	$\mathcal{D}_c$	$\mathcal{D}_{tc}$	$\mathcal{D}_n$	$\mathcal{D}_{nw}$	$\mathcal{D}_{nT}$
20 %	19	19	18	11	27
50 %	53	55	53	40	64
100 %	82	82	82	71	85
0.05	80	83	81	69	86
out	2	2	0	0	0

Set $\mathcal{S}_{na}$					
ν	design method				
level	$\mathcal{D}_c$	$\mathcal{D}_{tc}$	$\mathcal{D}_n$	$\mathcal{D}_{nw}$	$\mathcal{D}_{nT}$
20 %	11	10	8	6	14
50 %	43	41	41	27	50
100 %	73	72	72	56	77
0.05	68	66	67	49	74
out	2	1	0	0	0

Table 5.7: Experiment 5.6. Robust and nominal design of a filtering estimator. Left table: nice–systems $\mathcal{S}_{ni}$. Right table: nasty–systems $\mathcal{S}_{na}$. Percentage of filters resulting in a performance degradation less than ν, as compared to the optimal attainable performance for the system for which they are designed. The ν–levels are indicated in the first column. First three rows: percentage degradation of performance. Fourth row: absolute degradation of performance. Last row: percentage of outliers. Each set is composed by 40 systems. Number of design cases: 200. SNR = 13dB. DATA = 50.

performance improvement as compared to the use of the cautious filter. Note, however, that the tuning has been chosen from another experiment, so other values of the tuning parameters could result in a better performance. This latter fact indicates that the use of the robust method may be further complicated by the search for a best tuning. See also the previous experiment.

As compared to the use of a direct method, the outliers are eliminated. See the previous section. The outliers related to the robust methods are caused by numerical problems when solving the spectral factorization (5.6.9). It turned out that the polynomials B_1, A_1, M_1 and N_1 of the SOMD may often have zeros close to the unit circle. Moreover, the polynomials A_1 and N_1 have double zeros due to the series expansion from which they are obtained. See [Öhrn95]. These properties may cause numerical problems, and regularization of the algorithm may be required.

A more detailed analysis of the experiment outcome reveals interesting similarities with the results of [StAh93]. The aim of the robust filter is to improve the average performance (that is actually the criterion for which it is designed in Case 1), by reducing the occurrence of cases with a poor performance. After elimination of the outliers induced by numerical problems, the average performance of the methods resulted, for systems in the set $\mathcal{S}_{ni}$, as

$$\begin{aligned}\bar{V}_W &= 0.056\\ \bar{V}_c &= 0.088\\ \bar{V}_{tc} &= 0.088\\ \bar{V}_n &= 0.091\\ \bar{V}_{wn} &= 0.106 \quad ,\end{aligned}$$

while the worst–case performances were

$$\max V_c = 0.216$$

$$
\begin{aligned}
\max V_{tc} &= 0.217 \\
\max V_{n} &= 0.251 \\
\max V_{wn} &= 0.524 \ .
\end{aligned}
$$

It is interesting to note that $\mathcal{D}_{nT}$ results in the best design, and $\mathcal{D}_{nw}$ in the worst one: the loss of performance for methods other than $\mathcal{D}_{nT}$ seems to be mainly determined by errors in the noise model. In that respect it appears worthwhile to estimate the noise model, even when the number of data is small.

Experiments were conducted also for the case of smoothing lag $m = 4$. The outcome was similar to that of the present example.

In summary, the robust method has resulted in only a small improvement of performance as compared to the use of a nominal design. Note, also, that in this experiment the order of the estimated models was held fixed for all the design cases. Model validation could further reduce the utility of robustification. We interpret this result as being determined by the following cause:

> The estimation of models is rather efficient, also in the case of relatively few and noisy data. Model errors obtained after only a rough model validation, are in the range for which the use of nominal filters does not largely degrade the optimal performance. In Section 2.3 it was shown that optimal filters of high order can be effectively approximated by suboptimal filters of low order, say 4–6th. Hence, the optimal performance does, in general, not seem to be particularly sensitive. Perturbations of the optimal filters of the same magnitude as those caused by the use of an estimated model do almost never cause a large loss of performance as compared to the optimal one.

It should be noted that the validity of these conclusions is, of course, restricted to the design scenario considered in the experiment: data–based design of *linear deconvolution estimators* for *linear and time–invariant systems.* The cautious design has proven to be of considerable utility as compared to nominal designs for other design problems, such as DFE design and feedforward control, and for other causes of model uncertainty, such as time–variations or parametric uncertainty in physical modeling, see [StAhLi93], [LiAhSt93], [StÖhAh95], [Öhrn95]. ■

Finally, it is important to note that the indirect methods result in a performance statistics comparable to that obtained by the direct method for systems in the set $\mathcal{S}_{na}$, and slightly worse for systems in the set $\mathcal{S}_{ni}$. A notable difference is, however, that the outliers are eliminated. See Experiment 5.2. In the next section we will investigate in more detail the use of nominal indirect methods for filter design.

5.7 Indirect Methods

In this section we will further study the use of indirect methods for the design of a linear deconvolution filter. The nominal design and the two indirect methods introduced in

Section 2.4 will be considered. In the previous section the robust method resulted in only a small improvement of performance as compared to the use of a nominal method. In Section 5.3, the performance of the direct method turned out to be heavily influenced by the amount of data available for the design. With an adequate amount of data (DATA = 300), the direct method provided a rather good design in general, while frequent outlier situations appeared with fewer available data. There are therefore reasons to consider the use of the indirect methods, in particular when only short data sequences (DATA = 50) are available.

We will conduct four experiments.

In the first experiment we will investigate whether the two–stage indirect method introduced in Section 2.4, the *filtered nominal method*, can result in a performance improvement as compared to the use of a nominal design, as expected from the theoretical analysis. The optimal filter will be assumed to be known, and the channel model will be estimated from data, filtered through the inverse of the whitening filter. Under these conditions, the two–stage method results, in fact, in a general improvement of performance.

In Experiment 5.8 and Experiment 5.9, the indirect methods will be applied to the same design scenario as in Experiment 5.2, for the design of a filtering estimator ($m = 0$), and in Experiment 5.3, for the design of a smoother ($m = 4$). It will be shown that the choice of the model orders is crucial for obtaining a good performance. If appropriate model orders are selected, the nominal indirect design outperforms the direct method. The use of the two–stage method, as compared to Experiment 5.7, suffers from the use of an incorrect filter (the optimal whitening filter is not available) for the filtering of the data, and results in poor performances. The occurrence of outliers is, in general, negligible for all the investigated methods.

In Experiment 5.10 we will investigate the use of the *spectrum method*. It turns out that the performance is very poor in the case of short data sequences.

In summary, it is rather difficult to improve upon the performance statistics attained by the direct method, both for short and long data sequences. Nominal indirect designs can provide a better statistics only if well validated models are utilized. The selection of a good model order seems to be more critical in the pure filtering case than in the case of smoothing. Improvements of the identification algorithms, especially those for model validation, rather than the use of complicated design methods, seem to indicate the main requirement for a successful use of indirect methods. That issue represents as well the major direction for further research on the subject.

Experiment 5.7 Is the two–stage indirect method a sensible approach?

In Section 2.4, it was argued that filtering of the data through the inverse of the optimal whitening filter for estimating the channel model could result in an appropriate model for nominal design. In this experiment we investigate the use of the two–stage method when the optimal whitening filter is available. The design method is denoted by

$$\mathcal{D}_f \quad .$$

Experiments were conducted on 20 systems from each of the two sets, $\mathcal{S}_{ni}$ and $\mathcal{S}_{na}$ introduced in Section 5.4. Colored noises were used. The signal to noise ratio at the channel output was set to SNR = 13dB. The design of a filtering estimator was considered. For each system configuration, models were estimated 5 times, with different input and noise realizations. The model orders were chosen as

$$nG = 5 \ , \ nH = 3 \quad .$$

The models were estimated in two different ways. One was the standard estimation, which lead to the nominal design ($\mathcal{D}_n$). See the Experiment 5.6 in Section 5.6. In the second model, the channel model was obtained after filtering the data through the inverse of the optimal whitening filter. Then the noise model was estimated by fitting an ARMA model to the resulting model residual.

The designs were carried out for two different data lengths, DATA = 50 and DATA = 300, respectively.

The filtered method $\mathcal{D}_f$ outperformed the standard nominal design, but not with significant improvements. For short data sequences, out of the 100 design cases considered in the experiment, the use of the $\mathcal{D}_f$ method resulted in a better performance than the use of the $\mathcal{D}_n$, 62 times for systems in the set $\mathcal{S}_{ni}$ and 55 for systems in the set $\mathcal{S}_{na}$, respectively. The improvement increased for both sets when more data were available, and the above figures resulted as 70 and 63, respectively.

A possible explanation of the result could be that the transient of the filter limits the successful application of the two–stage method. The method does, however, seem promising enough to pursue further evaluation ■

Experiment 5.8 Indirect Methods. Filtering Estimator.

In this experiment we consider the same design scenario as in the Experiment 5.2, where the use of a direct method for the design of a filtering estimator was investigated. The design is carried out for the short data length, DATA = 50.

Four indirect methods are considered. The first two, $\mathcal{D}_n$ and $\mathcal{D}_{nw}$, are the standard nominal designs introduced in the Experiment 5.6. In the first design, an ARMA model for the noise is estimated, while in the second one the noise is modeled as being white. The inverse of the whitening filter of the filter designed with $\mathcal{D}_n$ is then utilized for a new estimation of the channel model, as in the Experiment 5.7. The new models so obtained result in the other two designs, $\mathcal{D}_f$ and $\mathcal{D}_{fw}$, respectively.

When estimating models, the model orders must be chosen. In Section 4.5, three approaches to model validation were distinguished. In the experiment all cases were considered. In the first approach, the model orders were held fixed to the values

$$nG = 5 \ , \ nH = 3$$

for all the designs. In the second approach, models were validated with the FPE criterion. See Section 4.5. In order to assess whether a good model validation is important for

Set $\mathcal{S}_{ni}$												
ν	5 + 3				FPE				z–val			
level	$\mathcal{D}_n$	$\mathcal{D}_f$	$\mathcal{D}_{nw}$	$\mathcal{D}_{fw}$	$\mathcal{D}_n$	$\mathcal{D}_f$	$\mathcal{D}_{nw}$	$\mathcal{D}_{fw}$	$\mathcal{D}_n$	$\mathcal{D}_f$	$\mathcal{D}_{nw}$	$\mathcal{D}_{fw}$
20 %	18	5	13	3	13	7	11	7	41	18	27	13
50 %	52	23	43	16	43	21	34	22	78	46	60	39
100 %	79	45	71	42	75	43	66	38	94	63	83	59
0.05	78	41	72	35	78	40	66	35	97	66	86	59
out	0	2	0	2	0	2	0	2	0	1	0	1

Set $\mathcal{S}_{na}$												
ν	5 + 3				FPE				z–val			
level	$\mathcal{D}_n$	$\mathcal{D}_f$	$\mathcal{D}_{nw}$	$\mathcal{D}_{fw}$	$\mathcal{D}_n$	$\mathcal{D}_f$	$\mathcal{D}_{nw}$	$\mathcal{D}_{fw}$	$\mathcal{D}_n$	$\mathcal{D}_f$	$\mathcal{D}_{nw}$	$\mathcal{D}_{fw}$
20 %	12	3	6	1	6	2	3	1	14	4	5	1
50 %	44	15	32	11	25	14	17	9	51	16	30	12
100 %	72	33	64	27	53	28	43	26	81	39	66	31
0.05	66	25	56	20	41	21	34	16	72	26	55	23
out	1	2	0	2	1	3	0	3	0	1	0	1

Table 5.8: Experiment 5.4: indirect method, filtering estimator. Upper table: nice–systems $\mathcal{S}_{ni}$. Lower table: nasty–systems $\mathcal{S}_{na}$. Percentage of filters resulting in a performance degradation less than ν, as compared to the optimal attainable performance for the system for which they are designed. The ν–levels are indicated in the first column. First three rows: percentage degradation of performance. Fourth row: absolute degradation of performance. Last row: percentage of outliers. Number of design cases: 200. SNR = 13dB. DATA = 50.

the design, we also introduced a third strategy, where a sequence of input–output data *z–val* of length $N_{val} = 500$, not used for the model estimation, was available for each system configuration. For increasing model orders, the estimated models were applied on the cross–validation data, and the model order that minimized the corresponding loss–function was therefore chosen. Note that the loss–function of the identification algorithm and not the performance of the obtained filter on the validation data was utilized as validation criterion. The experiment outcome is shown in Table 5.8, which should be compared to Table 5.2 in the Experiment 5.2. See also the explanation of the table there given.

Discussion. From Table 5.8 and Table 5.2, three different aspects are apparent:

- The performance statistics attained on the same design scenario by the use of the direct method, see Table 5.2 in the Experiment 5.2, reveals to be difficult to improve. Most variants of the indirect methods perform worse than the use of directly tuned filters of appropriate order.

- The role of model validation when using indirect methods turns out to be instrumental for obtaining a successful design.

- Indirect methods are in general more robust than direct methods in terms of outliers, as was already noted in the Experiment 5.3.

Set $\mathcal{S}_{ni}$												
ν	5 + 3				FPE				z–val			
level	$\mathcal{D}_n$	$\mathcal{D}_f$	$\mathcal{D}_{nw}$	$\mathcal{D}_{fw}$	$\mathcal{D}_n$	$\mathcal{D}_f$	$\mathcal{D}_{nw}$	$\mathcal{D}_{fw}$	$\mathcal{D}_n$	$\mathcal{D}_f$	$\mathcal{D}_{nw}$	$\mathcal{D}_{fw}$
20 %	11	4	4	1	10	4	3	2	26	12	6	1
50 %	49	22	26	9	46	25	26	10	78	38	35	16
100 %	81	39	55	28	76	45	55	33	97	63	71	48
0.05	76	37	47	18	71	40	47	26	94	59	61	36
out	0	0	0	1	0	1	0	1	0	1	0	1
Set $\mathcal{S}_{na}$												
ν	5 + 3				FPE				z–val			
level	$\mathcal{D}_n$	$\mathcal{D}_f$	$\mathcal{D}_{nw}$	$\mathcal{D}_{fw}$	$\mathcal{D}_n$	$\mathcal{D}_f$	$\mathcal{D}_{nw}$	$\mathcal{D}_{fw}$	$\mathcal{D}_n$	$\mathcal{D}_f$	$\mathcal{D}_{nw}$	$\mathcal{D}_{fw}$
20 %	12	3	5	2	10	5	3	3	21	11	9	6
50 %	49	20	26	10	38	22	18	14	68	36	34	20
100 %	78	39	57	30	65	42	50	34	90	57	64	42
0.05	66	30	45	16	53	29	33	22	83	45	56	28
out	0	1	0	1	1	0	0	2	0	1	1	1

Table 5.9: Experiment 5.8: indirect method, smoothing lag $m = 4$. Upper table: nice–systems $\mathcal{S}_{ni}$. Lower table: nasty–systems $\mathcal{S}_{na}$. Percentage of filters resulting in a performance degradation less than ν, as compared to the optimal attainable performance for the system for which they are designed. The ν–levels are indicated in the first column. First three rows: percentage degradation of performance. Fourth row: absolute degradation of performance. Last row: percentage of outliers. Number of design cases: 200. SNR = 13dB. DATA = 50.

It can be the case that the use of a fixed model order turns out to be almost as good as the use of good model validation. That was the case for systems in the set $\mathcal{S}_{na}$. On the contrary, a good model validation procedure may very well result in a large improvement, as for the systems in $\mathcal{S}_{ni}$. If the model orders are correctly chosen, the nominal indirect method $\mathcal{D}_n$ outperforms the direct method, both in terms of average performance and number of outliers [6]. A low occurrence of outliers is observed in general, also for poor choices of model orders. Estimation of an ARMA model for the noise always resulted in a better performance than the use of a white noise model.

In Table 5.8, two rather clear defeats can also be observed. The FPE criterion turns out to be a rather poor tool for validation, for short data sets. The use of the two–stage method with a filter different from the optimal one does not fulfill the expectations raised in the previous experiment. In general, it leads to very poor performances. Errors in the filter used for the filtering of the data seem to have a strong impact on the final performance.

The results of this experiment seem to confirm the interpretation we gave in the Experiment 5.6 about the cause for the small improvement of performance obtained with the use of the robust method as compared to that of the nominal design:

[6] The comparison of the performance statistics attained by using models cross–validated with the sequence *z–val* to the results of Table 5.2 is not a fair comparison, since a corresponding validation of the filters obtained with the direct methods was not carried out.

> The estimation of (validated) models turns out to be effective for the design of a deconvolution filter, also in the case of relatively noisy and few data. Instrumental is the validation of models for nominal design, rather than the use of more complicated design methods

■

Experiment 5.9 Indirect Methods. Smoothing with $m = 4$.

In this experiment we consider the same design scenario as in the Experiment 5.3, where the use of a direct method for the design of a smoother was investigated. The design is carried out for the short data length, DATA = 50.

The same design methods and the same strategies for choosing the model orders used in the previous experiment are now applied for the design of smoothers. The smoothing lag is set to the value $m = 4$.

The experiment outcome is shown in Table 5.9 and it should be compared to Table 5.3.

Discussion. Significant differences in the performance statistics as compared to the case of filtering are not observed. Note, however, that in the case of smoothing, the performance statistics seems less sensitive to the model order. Moreover, the performance statistics attained with $\mathcal{D}_n$ by the fixed choice of model orders, which is far from being the best as shown by the use of well validated models, is somewhat better than for the direct method, with the further advantage of the elimination of outliers.

The use of the indirect designs seems more suitable for the design of a smoother than for the design of a filtering estimator, for which the direct method performed effectively without requiring an elaborate model validation strategy ■

Experiment 5.10 The spectrum method.

The spectrum method was introduced in Section 2.4. Its use is investigated in this experiment. Unfortunately, the method revealed to require a rather large amount of data for providing successful designs. The estimation of an ARMA model for obtaining the innovation model of the output signal turns out to be rather ineffective in situations with short data sequences, and other algorithms for spectral estimation should be considered, see, for instance, [Kay88]. It is worth mentioning that poles and zeros of the estimated ARMA model were very often close to the unit circle. In order to limit the sensitivity of the resulting design, a constraint on their magnitude, so that it was never larger than 0.92, has been imposed after a model was obtained.

We only report the results obtained for systems in the set $\mathcal{S}_{ni}$, corresponding to the design of a filtering estimator for the design scenario of Experiment 5.2. They are given in Table 5.10 ■

DATA = 50						
ν	filter order					
level	3	4	5	6	7	8
20 %	1	0	0	0	0	0
50 %	2	1	0	0	0	0
100 %	11	3	1	2	0	2
0.05	8	2	2	1	0	0
out	1	1	2	2	4	2

DATA = 300						
ν	filter order					
level	3	4	5	6	7	8
20 %	7	5	3	2	1	1
50 %	52	43	43	28	26	15
100 %	82	81	80	72	67	60
0.05	87	87	82	72	69	54
out	0	0	0	0	0	0

Table 5.10: Experiment 5.9: spectrum method, filtering estimator. Nice–systems $\mathcal{S}_{ni}$. Percentage of filters resulting in a performance degradation less than ν, as compared to the optimal attainable performance for the system for which they are designed. The ν–levels are indicated in the first column. First three rows: percentage degradation of performance. Fourth row: absolute degradation of performance. Last row: percentage of outliers. Design cases: 200. SNR = 13dB.

5.8 Indirect Methods for DFE design

In this section we shall study the use of indirect methods for the design of decision feedback equalizers (DFE). A theoretical analysis of suboptimal design of DFEs was carried out in Chapter 3. In Section 3.3, the multiple prediction method was proposed as an alternative to the nominal design. In Section 3.4, it was shown that the minimization of the variance of the output predictions errors can lead to an optimal design for a given filters structure. The reader may wish to review those sections before proceeding.

We will investigate two aspects of the data based design of DFEs:

- whether the use of the multiple prediction method can effectively improve upon the performances attained by the nominal design,
- the utility for the design of the minimization of the variance of the output predictions error as compared to the use of other criteria for model estimation.

A note on direct methods. In Section 1.4, it was pointed out that, for the design of IIR DFEs, the use of the direct method is more involved than for the design of linear deconvolution filters. First, it is not clear how to choose the orders of the feedforward and feedback filters, and several combinations of filter orders need to be considered. Second, the design leads to a multivariable identification problem, with one output and two inputs, which is more difficult than the single output case. The multivariable identification problem can, however, be reduced to a SISO estimation problem, by estimating the filters of the DFE in the model equation, see (1.4.10),

$$\left[1 + q^{-1}F_b(q^{-1})\right] u(t-m) = F_f(q^{-1})y(t) + z(t) \quad . \tag{5.8.1}$$

As compared to the use of a standard output error method, modifications of the gradient of the loss–function are required for the estimation of the filters $F_f(q^{-1})$ and $F_b(q^{-1})$

in (5.8.1). The use of the direct method is expected to result in poor performances in situations with a small number of data, since a large number of parameters is to be estimated. The experiments in the previous sections showed that the number of parameters estimated in a successful design is strongly constrained by the available data. The same considerations as for the direct method apply when predictor filters are estimated directly on the data. Consideration of the above alternatives to the indirect methods is left for further research ■

We will report the outcome of two experiments as a summary for the typical behavior of the two indirect methods. The experiments refer to the design of a DFE for filtering, and are conducted on the same design scenario as the Experiments 5.2 and 5.7, with the only modification that the input is now a binary signal. In Experiment 5.11, models with fixed orders are estimated, while in Experiment 5.12, a data sequence is available for cross–validation. No significant differences in the behavior were observed in the case of smoothing.

The conclusions that can be drawn from the experiment are the following:

- as compared to the problem of designing a linear deconvolution filter, the design of a DFE scheme reveals to be more difficult, as larger relative deviations from the optimal performance are in general reported for the design methods,
- the use of the multiple prediction method improves upon the performance attained by a nominal design, but not substantially,
- strategies for the optimization of the modeling stage seem to require large data sets to be meaningful and the design appears to be mainly determined by the variance error and by possible limitations of the identification algorithm (e.g. convergence of the parameters estimate to local, but not global, minima of the loss–function).

A good model validation is not sufficient for obtaining reliable designs, and the estimation of models should be improved and possibly reconsidered. For instance, the first impulse response coefficients of the channel and the noise correlation computed for the first few lags may represent the crucial information for an effective design, rather than the estimated transfer functions.

Improvements of the identification algorithms, both for parameters estimation and model validation, indicate the major direction for further research on data–based design of IIR DFEs.

Experiment 5.11 DFE for filtering. Fixed order models.

In this experiment we consider the design of a DFE for filtering ($m = 0$) under the same design scenario as in Experiments 5.2 and 5.7, with the only modification that the input is now a binary signal, with equally likely symbols $\{-1, 1\}$. The optimal MSE performance attainable in the design cases, assuming correct past decisions, satisfies

$$V_{opt} \leq 0.07$$

and has an average value of

$$\bar{V}_{opt} = 0.025 \quad .$$

The DFE is designed with the nominal design ($\mathcal{D}_n$) and with the multiple prediction method, respectively. The latter method is denoted by

$$\mathcal{D}_p \quad .$$

Three different versions of each method are considered, depending on what noise model is utilized in the design equations. The first version, $\mathcal{D}_n$ and $\mathcal{D}_p$, is based on an estimated ARMA model. The second version, $\mathcal{D}_{nw}$ and $\mathcal{D}_{pw}$, is based on the noise modeled as being white. The third version, $\mathcal{D}_{nT}$ and $\mathcal{D}_{pT}$, is based on the correct noise model. The reason for considering the latter version is to investigate the quality of the channel model. See also the discussion in the Experiment 5.6.

Models are estimated with fixed model orders

$$nG = 5 \ , \ nH = 2$$

for all the design cases, by the two different strategies illustrated in Section 5.3. The first estimation strategy is the minimization of the variance of the one step ahead prediction error of the output signal. That strategy was shown to provide, in the filtering case, the best design for a given filter structure. The second strategy is the two–step standard estimation.

The designs were carried out for two data lengths, DATA = 50 and DATA = 300, respectively.

The experiment outcome is shown in Table 5.11, which refers to systems in the set $\mathcal{S}_{ni}$. In the fifth row, corresponding to the label "burst", the percentage of design cases where the occurrence of decision errors is rather likely is reported. A likely error burst case is defined as a DFE that results in a MSE performance, given correct past decision, of

$$V(S_T, \mathcal{D}) \ \geq \ 0.15$$

to which corresponds an estimation error with standard deviation

$$z(t)_{STD} = 0.39 \quad .$$

If the estimation errors are gaussian distributed, the probability of a bit error is then approximately equal to

$$P_e \ \approx \ 0.005 \quad .$$

For comparison with the design of linear deconvolution filters, also the percentage of outliers,

$$V(S_T, \mathcal{D}) \ \geq \ 1$$

is reported in the sixth row, corresponding to the label "out".

Discussion. By comparing Table 5.11 to Tables 5.2 and 5.9, the design of DFEs appears more difficult than the design of linear deconvolution filters. The design results in both large degradation of the optimal attainable performance and occurrence of outliers. The result could be interpreted as due to optimal DFE designs being rather sensitive:

DATA = 50												
ν	minimal prediction error						two–step estimation					
level	$\mathcal{D}_n$	$\mathcal{D}_{nw}$	$\mathcal{D}_{nT}$	$\mathcal{D}_p$	$\mathcal{D}_{pw}$	$\mathcal{D}_{pT}$	$\mathcal{D}_n$	$\mathcal{D}_{nw}$	$\mathcal{D}_{nT}$	$\mathcal{D}_p$	$\mathcal{D}_{pw}$	$\mathcal{D}_{pT}$
20 %	1	0	2	1	0	2	0	0	3	0	0	4
50 %	10	3	18	12	3	23	10	3	20	12	3	21
100 %	28	7	31	31	8	36	32	14	37	34	14	38
0.05	52	10	47	56	12	49	78	15	60	77	15	62
burst	30	58	42	29	57	40	12	37	25	12	36	24
out	8	10	27	5	7	8	4	11	10	5	11	6
DATA = 300												
ν	minimal prediction error						two–step estimation					
level	$\mathcal{D}_n$	$\mathcal{D}_{nw}$	$\mathcal{D}_{nT}$	$\mathcal{D}_p$	$\mathcal{D}_{pw}$	$\mathcal{D}_{pT}$	$\mathcal{D}_n$	$\mathcal{D}_{nw}$	$\mathcal{D}_{nT}$	$\mathcal{D}_p$	$\mathcal{D}_{pw}$	$\mathcal{D}_{pT}$
20 %	34	5	42	40	6	47	49	12	53	52	12	57
50 %	46	11	49	55	13	54	70	18	69	70	18	70
100 %	59	22	57	65	22	63	77	30	74	77	30	74
0.05	77	27	68	82	28	70	100	40	89	100	40	90
burst	16	27	28	12	24	22	0	1	4	0	1	3
out	3	4	16	2	2	3	0	0	2	0	0	0

Table 5.11: Experiment 5.11: DFE for filtering. Fixed order models. Nice–systems $\mathcal{S}_{ni}$. Percentage of filters resulting in a performance degradation less than ν, as compared to the optimal attainable performance for the system for which they are designed. The ν–levels are indicated in the first column. First three rows: percentage degradation of performance. Fourth row: absolute degradation of performance. Fifth row: percentage of cases with a likely error burst. Last row: percentage of outliers. Number of design cases: 200. SNR = 13dB.

model errors that cause only small performance degradations for linear deconvolution filters may result in useless designs for DFEs. The design is further constrained by the possible occurrence of error bursts due to the feedback path. It should however be pointed out that for a given system, the optimal attainable performance of a DFE is much better than that of an optimal linear deconvolution filter.

The strategy of minimizing prediction errors is clearly outperformed by the standard two–step estimation algorithm, independently of the number of data. The DFE design thus reveals to be mainly determined by the variance error of the estimated model and by possible limitations of the identification algorithm. For instance, the parameter estimate could converge to local, but not global, minima of the loss–function[7].

The multiple prediction method results in performances only slightly better than those attained by the nominal design.

The possible sensitivity of the optimal design seems to be confirmed by the performance statistics attained by the use of a correct noise description. Extreme cases are then more frequent, since both better "good performances" and worse "bad performances"

[7]While the standard two–step algorithm is provided with an effective initialization of the minimization search, the same is not available for the one–step algorithm. Convergence to local minima then seems more likely.

z–*val*, DATA = 50												
ν	minimal prediction error						two–step estimation					
level	$\mathcal{D}_n$	$\mathcal{D}_{nw}$	$\mathcal{D}_{nT}$	$\mathcal{D}_p$	$\mathcal{D}_{pw}$	$\mathcal{D}_{pT}$	$\mathcal{D}_n$	$\mathcal{D}_{nw}$	$\mathcal{D}_{nT}$	$\mathcal{D}_p$	$\mathcal{D}_{pw}$	$\mathcal{D}_{pT}$
20 %	5	3	9	6	3	11	8	3	14	8	4	14
50 %	20	12	29	22	12	29	28	12	37	30	12	39
100 %	39	18	40	41	18	42	52	22	51	54	23	51
0.05	72	18	53	75	20	54	92	28	66	94	28	68
burst	12	30	33	12	30	30	0	8	21	0	8	19
out	3	5	16	3	5	6	0	0	7	0	0	1

Table 5.12: Experiment 5.12: DFE for filtering. Cross–validated models. Nice–systems $\mathcal{S}_{ni}$. Percentage of filters resulting in a performance degradation less than ν, as compared to the optimal attainable performance for the system for which they are designed. The ν–levels are indicated in the first column. First three rows: percentage degradation of performance. Fourth row: absolute degradation of performance. Fifth row: percentage of cases with a likely error burst. Last row: percentage of outliers. Number of design cases: 200. SNR = 13dB.

are reported. That remaining effect is completely determined by the channel model, and suggests that better models for the channel are required for a more effective design. It is worth mentioning that the noise model plays an important role in the DFE design. For instance, the feedforward filter is formed by its inversion. In order to reduce the possible sensitivity introduced by zeros or poles of the noise model close to the unit circle, a constraint on their magnitude, so that it was never larger than 0.92, has been imposed after a model was obtained. When a noise model is estimated, a certain degree of self–robustification is introduced into the design. This explains the decreased occurrence of both very good and very bad results (compare $\mathcal{D}_n$ to $\mathcal{D}_{nT}$ in Table 5.11). It is also interesting to note that, despite the presumed sensitivity, it is still worthwhile to estimate an ARMA model of the noise, instead of assuming it to be white. This fact was also noted for the design of linear deconvolution filters.

An only slightly worse performance statistics was reported for systems in the set $\mathcal{S}_{na}$, without notable differences as compared to systems in the set $\mathcal{S}_{ni}$. That fact may suggest that systems that are "nasty" for the design of linear deconvolution filters, may not be so for the design of DFEs. It should be observed that in the latter case the channel dynamics is not inverted, and the design has a less direct dependence on the frequency characteristic of the channel.

Experiments were also conducted for the design of a DFE for smoothing with $m = 4$, on the same design scenario as considered in Experiments 5.3 and 5.9. The performance statistics did not shown significant differences as compared to the case of DFE for filtering ■

Experiment 5.12 DFE for Filtering. Cross–validated models.

In order to asses how much can be gained from a good model validation procedure, we repeat the previous experiment with designs based on cross–validated models. For each system configuration, a sequence of input–output data *z–val* of length $N_{val} = 500$, not used for the model estimation, was available. Then, for increasing model orders, the estimated models were applied on the cross–validation data, and the model order that minimized the corresponding loss–function was therefore chosen. Note that the loss–function of the identification algorithm and not the performance of the obtained filters on the validation data was utilized as validation criterion. Only models obtained by the standard model estimation algorithm were cross–validated. The same model orders were then utilized also with the minimal prediction error algorithm.

The designs were carried out for the short data length, DATA = 50.

The experiment outcome is shown in Table 5.12 and should be compared to Table 5.11.

Discussion. A good model validation may in itself not be sufficient for an effective design. Note, however, that the performance statistics is, in general, significantly improved as compared to the previous experiment. When models are estimated by the standard algorithm, the methods $\mathcal{D}_n$ and $\mathcal{D}_p$ provide quite acceptable performances, and cases with likely error bursts are eliminated.

It is interesting to note that the sensitivity of designs based on the correct noise model does not seem to be reduced by the model validation. Errors in the channel models turn out to be too large for a reliable design. Self–robustification can be introduced into the design by estimating an ARMA noise model, with, in the considered case, effective results. Compare $\mathcal{D}_{nT}$ to $\mathcal{D}_n$ and $\mathcal{D}_{pT}$ to $\mathcal{D}_p$ in Table 5.12. This fact may indicate that a cautious design, as suggested in [StAhLi93], would result in a more significant performance improvement for IIR DFEs than for the linear deconvolution estimators investigated in Experiment 5.6. Investigation of robust design of DFEs is a topic for further research ■

5.9 Sufficient Order Case

We conclude the simulation study by reporting in this section an experiment conducted in the sufficient order case. A set of 40 “nice” systems was generated as in Section 5.4. In each system, the channel and the noise innovation model had equal polynomial degrees

$$\begin{aligned} nG_T &= nb_T = na_T = 2 \\ nH_T &= nm_T = nn_T = 4 \quad . \end{aligned}$$

The design of a filtering estimator was considered, repeating for the systems above the Experiments 5.2 and 5.8 when designing a linear deconvolution estimator, and the Experiment 5.11 when designing a decision feedback equalizer. Refer to those experiments for the notations used in Table 5.13, where the experiment outcome is shown. Filters designed with the direct method and models for indirect design were chosen with the following orders:

$$nG = nH = nF = 2 \quad .$$

Linear Estimator			
ν	method		
level	$\mathcal{D}_{dir}$	$\mathcal{D}_n$	$\mathcal{D}_{nw}$
20 %	76	79	57
50 %	93	93	82
100 %	96	98	93
0.05	95	98	93
burst			
out	2	0	0

DFE				
ν	min–pe		2 step	
level	$\mathcal{D}_n$	$\mathcal{D}_p$	$\mathcal{D}_n$	$\mathcal{D}_p$
20 %	31	36	33	35
50 %	68	71	71	75
100 %	78	81	88	89
0.05	81	85	96	96
burst	15	11	1	1
out	5	3	0	0

Table 5.13: Experiment in the sufficient order case: design of a filtering estimator. Left table: linear deconvolution estimator. Right table: decision feedback equalizer. Percentage of filters resulting in a performance degradation less than ν, as compared to the optimal attainable performance for the system for which they are designed. The ν–levels are indicated in the first column. First three rows: percentage degradation of performance. Fourth row: absolute degradation of performance. Fifth row: percentage of cases with a likely error burst. Last row: percentage of outliers. Number of design cases: 200. SNR = 13dB. DATA = 50.

Conclusions

We have considered the use of system identification for the design of a deconvolution estimator. The design had to be carried out for an unknown system, with very limited a–priori information, not suitable in itself for proceeding with a design solution. Only input and noisy output data sequences, available during a short training period, were to be utilized.

A theoretical analysis on suboptimal solutions of the deconvolution problem in the presence of modeling errors has been carried out for the linear deconvolution estimator and for the decision feedback equalizer. The analysis has been conducted for discrete–time, linear, stable and time invariant, single–input, single–output systems with the output signal measured in the presence of additive measurement noise. Infinite impulse response structures for systems and models have been considered.

A simple expression for the sensitivity of the performance of the Wiener deconvolution estimator with respect to unstructured perturbations of the optimal filter has been obtained. The expression seems to be new. When designing suboptimal filters, the criterion to be considered in order to minimize the loss of performance is obtained as a result. The sensitivity of an optimal design can also be easily assessed. By means of computer simulations, it has been shown that optimal filters of high order can be effectively approximated by suboptimal filters of low–order, with only a small performance degradation. Strategies for approximate modeling to serve for nominal indirect design of linear deconvolution filters have been investigated and proposed.

A new principle for the MSE optimal design of DFEs has been obtained, which leads to a novel and effective method for suboptimal design. It has been shown that an optimal MSE design of DFEs can be seen as being based on two separated stages. In the first stage, optimal linear predictions of the output process $y(t)$ from past input–output values up to time $t-m-1$, where m is the smoothing lag, are calculated. Then, in the second, the estimate of $u(t-m)$ is obtained as the optimal linear mean square estimate based on the corresponding prediction errors. With this result, the problem of designing approximate DFE schemes is clarified and the role played by a constraint on the filter structure can be explained. A filter structure to be used for suboptimal design of DFEs has been proposed. Strategies for approximate modeling to serve for DFE design can also be investigated. An optimal strategy was found for the design of DFEs for filtering, that can also be useful for the design of DFEs for smoothing.

An extensive simulation study has been carried out to evaluate the theoretical analysis and to draw some general conclusion on differences in performance of various methods for model estimation and filter design. The simulation study provides a basic point of reference for further experiments and indicates directions for further research. A general conclusion is that models obtained by system identification are models “good enough” for the nominal design of linear deconvolution estimators, also when the identification experiment is performed with relatively few, and noisy, data, and the underlying system is complex. The use of an indirect nominal design turns out to result in performances comparable to those attained with a robust indirect method. The design of IIR DFEs is more difficult, and with short data sequences available for the design the resulting performance can be rather poor. Good model identification, and in particular good model validation, seems to be the crucial factor for a successful use of indirect methods. It is definitely more important than the introduction of more complicated methods in the filter design stage.

The two limiting factors to the use of system identification for design purposes, mentioned in the Introduction to this study, turned out not to constitute a serious limit for a successful design, even in situations with short (50 samples), and noisy (SNR = 13dB) data sequences, and channels with a high order transfer function (12 zeros and 12 poles) of various complexity. System identification shows indeed a great potential as a tool for an automated filter design based on data observation.

Appendix A

Proofs in Chapter 2

This Appendix contains the proofs of the theorems and the lemmas in Chapter 2.

A.1 Proof of Lemma 2.1

When using the Wiener filter, the estimation error (2.2.4) is given by

$$\begin{aligned} z(t) &= u(t-m) - \hat{u}(t-m) \\ &= \left[q^{-m} - F_W(q^{-1})G(q^{-1})\right] u(t) - F_W(q^{-1})v(t) \end{aligned} \tag{A.1.1}$$

where (2.2.1) has been used. Since $u(t)$ and $v(t)$ are assumed to be mutually uncorrelated, the spectrum of the estimation error is given by

$$S_z(e^{j\omega}) = \mid e^{-jm\omega} - F_W(e^{j\omega})G(e^{j\omega}) \mid^2 S_u(e^{j\omega}) + \mid F_W(e^{j\omega}) \mid^2 S_v(e^{j\omega}) \tag{A.1.2}$$

where

$$\begin{aligned} S_u(e^{j\omega}) &= \lambda_w \mid H_u(e^{j\omega}) \mid^2 \\ S_v(e^{j\omega}) &= \lambda_e \mid H(e^{j\omega}) \mid^2 \quad . \end{aligned}$$

The MSE criterion (2.2.5) can be rewritten in terms of the spectrum $S_z(e^{j\omega})$ as:

$$\begin{aligned} V_W &= \mathrm{E}\ z^2(t) = \frac{1}{\pi}\int_0^{\pi} S_z(e^{j\omega})d\omega \\ &= \frac{1}{\pi}\int_0^{\pi}\left\{\mid e^{-jm\omega} - F_W(e^{j\omega})G(e^{j\omega}) \mid^2 S_u(e^{j\omega}) + \mid F_W(e^{j\omega}) \mid^2 S_v(e^{j\omega})\right\} d\omega \end{aligned} \tag{A.1.3}$$

In the following, we will drop the arguments $e^{j\omega}$ and q^{-1} for simplicity of notation. Using (2.2.12) and (2.2.10), the spectrum of the output signal $y(t)$ is given by

$$S_y = \left[\mid G \mid^2 S_u + S_v\right] = \frac{\lambda_w}{\mid F_{wh} \mid^2} = \lambda_w \frac{\mid F_{sh} \mid^2}{\mid F_W \mid^2} \tag{A.1.4}$$

where F_{wh} is the whitening filter defined in (2.2.11) and F_{sh} is the shaping filter defined in (2.2.13). With the use of expressions (A.1.4), the MSE criterion (A.1.3) becomes

$$\begin{aligned} V_W &= \frac{1}{\pi}\int_0^{\pi}\left\{S_u - 2\mathrm{Re}[e^{jm\omega}GF_W]S_u + \mid GF_W \mid^2 S_u + \mid F_W \mid^2 S_v\right\} d\omega \\ &= \frac{1}{\pi}\int_0^{\pi}\left\{S_u + \lambda_w \mid F_{sh} \mid^2 - 2\mathrm{Re}[e^{jm\omega}GF_W S_u]\right\} d\omega \qquad \text{(A.1.5)} \end{aligned}$$

where Re[c] denotes the real part of a complex number c. With the use of (2.2.5)–(2.2.7), consider the term

$$\begin{aligned} q^m GF_W S_u &= q^m \frac{B}{A} Q_1 \frac{AN}{\beta} \lambda_w \frac{CC_*}{DD_*} = \lambda_w \frac{Q_1}{\beta DD_*}\left[q^m BNC_*C\right] \\ &= \lambda_w \frac{Q_1}{\beta DD_*}\left[r\beta Q_{1*} + q^{-1}D_*L\right] = \lambda_w r \frac{Q_1 Q_{1*}}{DD_*} + q^{-1}\lambda_w \frac{Q_1 L}{\beta D} \\ &= \lambda_w \mid F_{sh} \mid^2 + q^{-1}\lambda_w \frac{Q_1 L}{\beta D} \qquad \text{(A.1.6)} \end{aligned}$$

where (2.2.13) was used in the last equality. For a causal transfer function

$$f(q^{-1}) = \sum_0^{\infty} f_k q^{-k}$$

observe that

$$\frac{1}{\pi}\int_0^{\pi} \mathrm{Re}[f(e^{-j\omega})]d\omega = f_0 \quad . \qquad \text{(A.1.7)}$$

Hence, since the first impulse response coefficient is zero in the last term in (A.1.6), we obtain

$$\frac{1}{\pi}\int_0^{\pi} \mathrm{Re}[e^{jm\omega}GF_W S_u]d\omega = \lambda_w \frac{1}{\pi}\int_0^{\pi} \mid F_{sh} \mid^2 d\omega = \lambda_w \alpha \quad . \qquad \text{(A.1.8)}$$

Recall that by definition,

$$\frac{1}{\pi}\int_0^{\pi} S_u d\omega = \lambda_u = \lambda_w \rho_u \quad .$$

The lemma follows by inserting (A.1.8) into (A.1.5) ∎

A.2 Proof of Lemma 2.2

From Lemma 2.1 and (2.2.17), the optimal performance is given by

$$V_W = \lambda_w(1-\alpha) \ \geq \ 0 \qquad \text{(A.2.1)}$$

Apparently, the Wiener filter performance varies between the value $V_W = 0$, when the noise in (2.2.1) is not present and the channel transfer function $G(q^{-1})$ is minimum–phase, and the value $V_W = \lambda_w$, when the noise variance tends to infinity. In the latter case, the filter gain vanishes. For the whitening filter F_{wh} defined in (2.2.11), the following expression holds:

$$\mid F_{wh} \mid^2 = \frac{\lambda_w}{\mid G \mid^2 S_u + S_v} \qquad \text{(A.2.2)}$$

where $S_u \equiv \lambda_w$ in the considered case, with $H_u = 1$. The shaping filter F_{sh} defined in (2.2.13) is then given by

$$F_{sh} = \sqrt{r}Q_1 = \frac{g_0}{\sqrt{r}} \tag{A.2.3}$$

$$| F_{sh} |^2 = \alpha \ . \tag{A.2.4}$$

See Remark 2.1. With the use of (2.2.10), (A.2.1), (A.2.2) and (A.2.4), consider the following quantity:

$$\begin{aligned}\frac{1}{| G |^2} - | F_W |^2 &= \frac{1}{| G |^2} - \alpha \frac{\lambda_w}{| G |^2 \lambda_w + S_v} \\ &= \frac{1}{| G |^2 (| G |^2 \lambda_w + S_v)} \left[S_v + \lambda_w (1-\alpha) \mid G |^2 \right] \\ &= \frac{1}{| G |^2 (| G |^2 \lambda_w + S_v)} \left[S_v + V_W \mid G |^2 \right] \geq 0\end{aligned}$$

The stated inequality follows ■

A.3 Proof of Theorem 2.1

For an arbitrary stable deconvolution estimator $F(q^{-1})$, consider the expression (2.3.13)

$$F(q^{-1}) = F_W(q^{-1}) + \bar{F}_W(q^{-1})\tilde{\Delta}_F(q^{-1}) \tag{A.3.1}$$

where $F_W(q^{-1})$ is the Wiener filter, $\bar{F}_W(q^{-1})$ is the minimum phase Wiener filter defined in (2.3.7) and $\tilde{\Delta}_F(q^{-1})$ is the stable perturbation defined in (2.3.12).

The MSE performance of the filter $F(q^{-1})$ when applied to data generated by the system S is given by

$$\begin{aligned}V(S,F) &= \mathrm{E}\ z^2(t) = \frac{1}{\pi}\int_0^\pi S_z(e^{j\omega})d\omega \\ &= \frac{1}{\pi}\int_0^\pi \left\{ | e^{-jm\omega} - FG |^2 S_u + | F |^2 S_v \right\} d\omega \end{aligned} \tag{A.3.2}$$

where the argument $e^{j\omega}$ was dropped for simplicity of notation. By inserting (A.3.1) into (A.3.2), the MSE performance results as

$$\begin{aligned}V(S,F) &= \frac{1}{\pi}\int_0^\pi \left\{ | \left(e^{-jm\omega} - F_W G\right) - \bar{F}_W G \tilde{\Delta}_F |^2 S_u + | F_W + \bar{F}_W \tilde{\Delta}_F |^2 S_v \right\} d\omega \\ &= \frac{1}{\pi}\int_0^\pi \left\{ | e^{-jm\omega} - F_W G |^2 S_u + | F_W |^2 S_v \right\} d\omega \\ &\quad + \frac{1}{\pi}\int_0^\pi | \tilde{\Delta}_F |^2 \left(| G |^2 S_u + S_v \right) | \bar{F}_W |^2 d\omega \\ &\quad + 2\frac{1}{\pi}\int_0^\pi \mathrm{Re}\left\{ \tilde{\Delta}_F \left[- \left(e^{jm\omega} - F_{W_*} G_* \right) \bar{F}_W G S_u + F_{W_*} \bar{F}_W S_v \right] \right\} d\omega \end{aligned} \tag{A.3.3}$$

The first term of the sum in (A.3.3) corresponds to the performance V_W of the Wiener filter F_W. Consider the second term of the sum in (A.3.3). From (A.1.4),

$$\left(\mid G\mid^2 S_u + S_v\right) \mid \bar{F}_W \mid^2 = \lambda_w \mid F_{sh}\mid^2 \quad . \tag{A.3.4}$$

Consider the expression within square brackets in the third term of the sum in (A.3.3).

$$-\left(e^{jm\omega} - F_{W_*}G_*\right)\bar{F}_W G S_u + F_{W_*}\bar{F}_W S_v = \bar{F}_W F_{W_*}\left(\mid G\mid^2 S_u + S_v\right) - e^{jm\omega}\bar{F}_W G S_u \quad . \tag{A.3.5}$$

From (2.2.6) and the spectral factorization (2.2.7),

$$\begin{aligned} \bar{F}_W F_{W_*}\left(\mid G\mid^2 S_u + S_v\right) &= \frac{\bar{Q}_1 N A}{\beta}\frac{Q_{1_*}N_*A_*}{\beta_*}\frac{r\beta\beta_*}{AA_*NN_*DD_*}\lambda_w \\ &= \lambda_w r\frac{\bar{Q}_1 Q_{1_*}}{DD_*} \quad . \end{aligned} \tag{A.3.6}$$

From (2.3.7), (2.3.8) and the Diophantine equation (2.2.8), the last term of (A.3.5), with q substituted for $e^{jm\omega}$, can be expressed as

$$\begin{aligned} q^m \bar{F}_W G S_u &= q^m\frac{\bar{Q}_1 AN}{\beta}\frac{B}{A}\lambda_w\frac{CC_*}{DD_*} \\ &= \lambda_w\frac{\bar{Q}_1}{\beta DD_*}\left[q^m BNC_*C\right] \\ &= \lambda_w\frac{\bar{Q}_1}{\beta DD_*}\left[r\beta Q_{1_*} + q^{-1}D_*L\right] \\ &= \lambda_w r\frac{\bar{Q}_1 Q_{1_*}}{DD_*} + q^{-1}\lambda_w\frac{\bar{Q}_1 L}{\beta D} \quad . \end{aligned} \tag{A.3.7}$$

By inserting (A.3.6) and (A.3.7) into (A.3.5), the third term of (A.3.3) is shown to vanish:

$$\begin{aligned} &2\frac{1}{\pi}\int_0^\pi \text{Re}\,\left\{\tilde{\Delta}_F\left[-\left(e^{jm\omega} - F_{W_*}G_*\right)\bar{F}_W G S_u + F_{W_*}\bar{F}_W S_v\right]\right\}d\omega = \\ &= -2\frac{1}{\pi}\int_0^\pi \text{Re}\,\left[\tilde{\Delta}_F e^{-j\omega}\lambda_w\frac{\bar{Q}_1 L}{\beta D}\right]d\omega = 0 \end{aligned}$$

since the transfer function

$$q^{-1}\tilde{\Delta}_F\frac{\bar{Q}_1 L}{\beta D}$$

has the first coefficient of the Fourier transform equal to zero. Compare to (A.1.7). The proof follows ■

A.4 Invariance of the Deconvolution Performance

In Section 2.3, it was observed that an interesting feature of the Wiener filter shown by the expression (2.3.15) is that its performance is equally sensitive to both an increase or decrease of the filter magnitude. Apparently, the filters

$$\begin{aligned} F_1(q^{-1}) &= F_W(q^{-1})(1+\delta) \\ F_2(q^{-1}) &= F_W(q^{-1})(1-\delta) \end{aligned}$$

with a scalar $\delta > 0$, will result in the same performance degradation. The same holds when any stable perturbation $\Delta(q^{-1})$ is considered instead of the scalar δ, as shown in Theorem 2.1 and Remark 2.2.

A reasonable question is then whether suboptimal filters have the same property. The answer is positive only in particular cases. Those cases will be given by Lemma A.1. Lemma A.1 follows from the expression (A.4.3) given below.

Consider the use of a (stable) filter F on data generated by the system S. Denote its performance by

$$V_F \triangleq V(S, F) \quad .$$

With the use of a stable transfer function $\Delta(q^{-1})$, form the filters

$$F_+(q^{-1}) = F(q^{-1})\left[1 + \Delta(q^{-1})\right] \tag{A.4.1}$$

$$F_-(q^{-1}) = F(q^{-1})\left[1 - \Delta(q^{-1})\right] \tag{A.4.2}$$

and denote by

$$\begin{aligned} V_+ &\triangleq V(S, F_+) \\ V_- &\triangleq V(S, F_-) \end{aligned}$$

the corresponding performances. Then, the following holds

$$\frac{1}{\pi}\int_0^{\pi} \mid \Delta \mid^2 \mid F \mid^2 S_y d\omega = \frac{V_+ + V_- - 2V_F}{2} \quad . \tag{A.4.3}$$

In the expression (A.4.3), the transfer functions Δ and F can be tuned until the right-hand side is equal to one, then matching the output spectrum

$$S_y = \mid G \mid^2 S_u + S_v \quad .$$

The performance of the filter F_+ is given by

$$\begin{aligned} V_+ &= \frac{1}{\pi}\int_0^{\pi} \left\{\mid e^{-jm\omega} - F(1+\Delta)G \mid^2 S_u + \mid F(1+\Delta) \mid^2 S_v\right\} d\omega \\ &= \frac{1}{\pi}\int_0^{\pi} \left\{\mid e^{-jm\omega} - FG \mid^2 S_u + \mid F \mid^2 S_v\right\} d\omega \\ &+ \frac{1}{\pi}\int_0^{\pi} \mid \Delta \mid^2 \mid F \mid^2 \left[\mid G \mid^2 S_u + S_v\right] d\omega \\ &+ 2\frac{1}{\pi}\int_0^{\pi} \text{Re}\left\{\Delta \mid F \mid^2 \left[\mid G \mid^2 S_u + S_v\right] - \Delta e^{jm\omega} FGS_u\right\} d\omega \quad . \end{aligned} \tag{A.4.4}$$

Apparently, the first term in the sum corresponds to V_F. The performance of the filter F_- is given by

$$\begin{aligned} V_- &= \frac{1}{\pi}\int_0^{\pi} \left\{\mid e^{-jm\omega} - F(1-\Delta)G \mid^2 S_u + \mid F(1-\Delta) \mid^2 S_v\right\} d\omega \\ &= \frac{1}{\pi}\int_0^{\pi} \left\{\mid e^{-jm\omega} - FG \mid^2 S_u + \mid F \mid^2 S_v\right\} d\omega \\ &+ \frac{1}{\pi}\int_0^{\pi} \mid \Delta \mid^2 \mid F \mid^2 \left[\mid G \mid^2 S_u + S_v\right] d\omega \\ &- 2\frac{1}{\pi}\int_0^{\pi} \text{Re}\left\{\Delta \mid F \mid^2 \left[\mid G \mid^2 S_u + S_v\right] - \Delta e^{jm\omega} FGS_u\right\} d\omega \quad . \end{aligned} \tag{A.4.5}$$

Then, the expression (A.4.3) follows by adding together the expressions (A.4.4) and (A.4.5).

Based on the expressions (A.4.4) and (A.4.5), the following result is obtained.

Lemma A.1 Assume that the Wiener filter is minimum phase[1]. Set the transfer function Δ in (A.4.1), (A.4.2) to the value

$$\Delta(q^{-1}) = \delta > 0$$

with a scalar δ. The filter F in (A.4.1), (A.4.2) can be expressed by a multiplicative stable perturbation $\Delta_F(q^{-1})$ as

$$F(q^{-1}) = F_W(q^{-1})\left[1 + \Delta_F(q^{-1})\right] \tag{A.4.6}$$

where F_W is the Wiener filter of the system S. Then, the resulting filters $F_+(q^{-1})$ and $F_-(q^{-1})$ in (A.4.1), (A.4.2) will have the same performance when applied to the system S if and only if

$$\frac{1}{\pi}\int_0^\pi \left[\mid \Delta_F \mid^2 + \mathrm{Re}(\Delta_F)\right] \mid F_{sh} \mid^2 d\omega = 0$$

where F_{sh} is the shaping filter defined in (2.2.13).

Proof. From (A.4.4) and (A.4.5), the filters $F_+(q^{-1})$ and $F_-(q^{-1})$ will have the same performance when applied to the system S if and only if

$$\frac{1}{\pi}\int_0^\pi \mathrm{Re}\left\{\mid F \mid^2 \left[\mid G \mid^2 S_u + S_v\right] - e^{jm\omega} F G S_u\right\} d\omega = 0 \quad . \tag{A.4.7}$$

Recall that

$$\begin{aligned} F_W &= F_{sh} F_{wh} && \text{(A.4.8)} \\ \mid F_{wh} \mid^2 &= \frac{\lambda_w}{\mid G \mid^2 S_u + S_v} && \text{(A.4.9)} \\ q^m G F_W S_u &= \lambda_w \mid F_{sh} \mid^2 + \lambda_w q^{-1} \frac{Q_1 L}{\beta D} \quad . && \text{(A.4.10)} \end{aligned}$$

See (2.2.10), (2.2.11) and (A.1.6). By inserting (A.4.6) and (A.4.8)–(A.4.10) into (A.4.7),

$$\begin{aligned} 0 &= \frac{1}{\pi}\int_0^\pi \mathrm{Re}\left\{\mid F_{sh} \mid^2 \mid 1 + \Delta_F \mid^2 - \left[\mid F_{sh} \mid^2 + e^{-jm\omega}\frac{Q_1 L}{\beta D}\right][1 + \Delta_F]\right\} d\omega \\ &= \frac{1}{\pi}\int_0^\pi \mathrm{Re} \mid F_{sh} \mid^2 \left\{\mid 1 + \Delta_F \mid^2 - [1 + \Delta_F]\right\} d\omega \quad . && \text{(A.4.11)} \end{aligned}$$

The second equality in (A.4.11) follows because the term

$$q^{-1}\frac{Q_1 L}{\beta D}[1 + \Delta_F]$$

has the first impulse coefficient equal to zero. Compare to (A.1.7). The proof follows from (A.4.11) ■

[1] The general case involves only the use of a more cumbersome notations.

A.5 Proof of Lemma 2.3

In the case of pure filtering ($m = 0$) of a white input, the Wiener filter for the system S_T is given by, see Remark 2.1,

$$F_W(q^{-1}) = \frac{g_0}{\sqrt{r_T}} F_{whT}(q^{-1}) \tag{A.5.1}$$

where F_{whT} is the whitening filter defined in (2.2.11), g_0 is the first impulse response coefficient of the channel $G_T(q^{-1})$, and the scalar r_T is obtained by the spectral factorization (2.2.7). The nominal filter obtained from the model S is given by

$$F_n(q^{-1}) = \frac{\hat{g}_0}{\sqrt{r}} F_{wh}(q^{-1}) \tag{A.5.2}$$

with obvious meaning of the notation. The multiplicative perturbations $\Delta_F(q^{-1})$ and $\tilde{\Delta}_F(q^{-1})$ in (2.3.11) and (2.3.12) coincide, since the Wiener filter is minimum phase. From (2.3.12), by the use of (A.5.1) and (A.5.2):

$$\begin{aligned} \tilde{\Delta}_F(q^{-1}) &= F_W^{-1}(q^{-1}) \left[F_n(q^{-1}) - F_W(q^{-1}) \right] \\ &= \frac{\hat{g}_0}{\sqrt{r}} \cdot \frac{\sqrt{r_T}}{g_0} \cdot \frac{F_{wh}(q^{-1})}{F_{whT}(q^{-1})} - 1 \end{aligned} \tag{A.5.3}$$

By using (2.2.16),

$$\alpha_T = \frac{g_0^2}{r_T} \quad ; \quad \alpha = \frac{\hat{g}_0^2}{r} \tag{A.5.4}$$

and (2.2.12), the square magnitude of $\tilde{\Delta}_F(q^{-1})$ is then given by

$$\mid \tilde{\Delta}_F \mid^2 = \frac{\alpha}{\alpha_T} \cdot \frac{\mid G_T \mid^2 \lambda_w + \rho_T \mid H_T \mid^2}{\mid G \mid^2 \lambda_w + \rho \mid H \mid^2} + 1 - 2\mathrm{Re} \left[\frac{\hat{g}_0}{\sqrt{r}} \cdot \frac{\sqrt{r_T}}{g_0} \cdot \frac{F_{wh}}{F_{whT}} \right] \quad . \tag{A.5.5}$$

By inserting (A.5.5) into (2.3.15), the loss of performance caused by the use of the filter F_n instead of F_W is given by

$$\begin{aligned} \delta_V &= \frac{1}{\pi} \int_0^{\pi} \mid \tilde{\Delta}_F \mid^2 \left| \frac{g_0}{\sqrt{r_T}} \right|^2 d\omega \\ &= \alpha_T \frac{1}{\pi} \int_0^{\pi} \left\{ \frac{\alpha}{\alpha_T} \cdot \frac{\mid G_T \mid^2 \lambda_w + \rho_T \mid H_T \mid^2}{\mid G \mid^2 \lambda_w + \rho \mid H \mid^2} + 1 - 2\mathrm{Re} \left[\frac{\hat{g}_0}{\sqrt{r}} \cdot \frac{\sqrt{r_T}}{g_0} \cdot \frac{F_{wh}}{F_{whT}} \right] \right\} d\omega \\ &= \alpha \frac{1}{\pi} \int_0^{\pi} \frac{\mid G_T \mid^2 \lambda_w + \rho_T \mid H_T \mid^2}{\mid G \mid^2 \lambda_w + \rho \mid H \mid^2} d\omega + \alpha_T - 2 \frac{\hat{g}_0 g_0}{r} \end{aligned} \tag{A.5.6}$$

where the last equality follows, by using (A.5.4), from (A.1.7), since the first impulse response coefficient of

$$\frac{F_{wh}(q^{-1})}{F_{whT}(q^{-1})}$$

is, from (2.2.11),

$$\frac{\sqrt{r_T}}{\sqrt{r}} \quad .$$

The proof follows from (A.5.6) ■

Appendix B

Proofs in Chapter 3

This Appendix contains the proofs of the theorems and the lemmas in Chapter 3, and the derivation of the expressions used in the Examples 3.1 and 3.2. We will make an extensive use of the polynomial equations for obtaining optimal linear predictions of the output signal $y(t)$ from past input and output values. For the convenience of the reader, we will summarize the predictor design in the following section.

B.1 Polynomial Equations for Optimal Linear Prediction

For the results of this section, refer, for instance, to [AhSt94] or [Söd94].

Data are assumed to be generated by the linear, time–invariant and stable system

$$\begin{aligned} y(t) &= G(q^{-1})u(t) + v(t) \\ v(t) &= H(q^{-1})e(t) \\ \mathrm{E}e^2(t) &= \rho \end{aligned} \tag{B.1.1}$$

where the noise process $v(t)$ is expressed in its innovation form. The transfer function $H(q^{-1})$ is then minimum phase. That is not required for the transfer function $G(q^{-1})$. The input process $u(t)$ is zero mean and white, with unit variance. The input is uncorrelated with the innovation process $e(t)$. The transfer functions in (B.1.1) are expressed by polynomials in the backward shift operator q^{-1}:

$$\begin{aligned} G(q^{-1}) &= \frac{B(q^{-1})}{A(q^{-1})} = \sum_{k=0}^{\infty} g_k q^{-k} \\ H(q^{-1}) &= \frac{M(q^{-1})}{N(q^{-1})} = 1 + \sum_{k=1}^{\infty} h_k q^{-k} \quad . \end{aligned}$$

The polynomial degrees are denoted nb, na, etc.

Using stable, linear and causal filters $\hat{F}_{y,i}(q^{-1})$ and $\hat{F}_{u,i}(q^{-1})$, the following linear prediction of $y(t)$ can be formed based on the past values of the input and output up to time $t-i$

$$\hat{y}(t|t-i) = \hat{F}_{y,i}(q^{-1})y(t-i) + \hat{F}_{u,i}(q^{-1})u(t-i) \quad . \tag{B.1.2}$$

The corresponding prediction error is

$$\tilde{y}(t|t-i) = y(t) - \hat{y}(t|t-i) \quad . \tag{B.1.3}$$

The problem is to find the predictor filters $\hat{F}_{y,i}(q^{-1})$ and $\hat{F}_{u,i}(q^{-1})$ that minimize the variance of the prediction error

$$V(\hat{F}_{y,i}, \hat{F}_{u,i}) = \mathrm{E}\ \tilde{y}^2(t|t-i) \tag{B.1.4}$$

over the set of all stable, linear and causal filters. The optimal filters, denoted as $F_{y,i}(q^{-1})$, $F_{u,i}(q^{-1})$, are given by the following theorem.

Theorem B.1 Design of optimal linear predictors. Solve, with respect to $B_i(q^{-1})$, $P_i(q^{-1})$, $M_i(q^{-1})$, $T_i(q^{-1})$, the following polynomial equations

$$B(q^{-1}) = P_i(q^{-1})A(q^{-1}) + q^{-i}B_i(q^{-1}) \tag{B.1.5}$$
$$M(q^{-1}) = T_i(q^{-1})N(q^{-1}) + q^{-i}M_i(q^{-1}) \tag{B.1.6}$$

where[1]

$$\begin{aligned} nb_i &= \max(nb, na) - 1 \\ nm_i &= \max(nm, nn) - 1 \end{aligned}$$

and[2]

$$P_i(q^{-1}) = p_1 + p_2 q^{-1} + \ldots + p_i q^{-i+1} \tag{B.1.7}$$
$$T_i(q^{-1}) = 1 + t_2 q^{-1} + \ldots + t_i q^{-i+1} \quad . \tag{B.1.8}$$

Then the filters $\hat{F}_{y,i}(q^{-1})$ and $\hat{F}_{u,i}(q^{-1})$ in (B.1.2), that minimize the MSE criterion (B.1.4) over the set of all stable, linear and causal filters, are given by the following expression

$$\boxed{\begin{aligned} F_{y,i}(q^{-1}) &= \frac{M_i(q^{-1})}{M(q^{-1})} \\ F_{u,i}(q^{-1}) &= \frac{M(q^{-1})B_i(q^{-1}) - M_i(q^{-1})B(q^{-1})}{M(q^{-1})A(q^{-1})} \end{aligned}} \tag{B.1.9}$$

The minimal prediction error is given by

$$\boxed{\tilde{y}_o(t|t-i) = P_i(q^{-1})u(t) + T_i(q^{-1})e(t)} \tag{B.1.10}$$

[1]The expressions are obtained by setting the number of equations equal to the number of unknown coefficients in (B.1.5) and (B.1.6). If a polynomial order results to be negative, then the corresponding polynomial is set to zero.

[2]The notations for the polynomials P_i and T_i differ from the standard notations, since the leading coefficients have subscript equal to 1 instead of 0. The reason is that in this way, the last coefficient has subscript equal to i, where i corresponds to the number of steps ahead in the prediction.

Proof. The filters (B.1.9) are optimal *if and only if* the estimate is orthogonal to the prediction error:

$$\mathrm{E}\ \tilde{y}(t|t-i)\hat{y}(t|t-i) = 0 \quad .$$

See, e.g. [Söd94]. Insert (B.1.5)–(B.1.6) into (B.1.1). Apparently:

$$\begin{aligned} y(t) &= \left[P_i + q^{-i}\frac{B_i}{A}\right] u(t) + \left[T_i + q^{-i}\frac{M_i}{N}\right] e(t) \\ &= [P_i u(t) + T_i e(t)] + \frac{B_i}{A} u(t-i) + \frac{M_i}{N}\frac{N}{M}\left[y(t-i) - \frac{B}{A}u(t-i)\right] \\ &= [P_i u(t) + T_i e(t)] + \left[\frac{B_i}{A}u(t-i) - \frac{M_i B}{AM}u(t-i) + \frac{M_i}{M}y(t-i)\right] \quad . \end{aligned}$$

The second equality can be utilized because $M(q^{-1})$ is assumed stable. The term

$$\frac{B_i M - M_i B}{MA} u(t-i) + \frac{M_i}{M} y(t-i) \tag{B.1.11}$$

depends only on past values of the processes $u(t)$ and $e(t)$ up to time $t-i$. Since both $u(t)$ and $e(t)$ are white, it is uncorrelated with the term

$$P_i u(t) + T_i e(t) = p_1 u(t) + \ldots + p_i u(t-i+1) + e(t) + \ldots + t_i e(t-i+1) \quad . \tag{B.1.12}$$

Hence, the expression (B.1.11) constitutes the optimal prediction, and the expression (B.1.12) is the minimal prediction error. The theorem follows ∎

B.2 Proof of Theorem 3.1

The proof is divided in two parts. First the optimal weight vector $\bar{\gamma}$ given by (3.2.29) is obtained, by minimizing the MSE criterion with respect to the vector γ. Then, it is shown that the corresponding estimate $\hat{u}(t-m)$ in (3.2.28) is, indeed, the optimal estimate among all linear MSE estimate of $u(t-m)$ which are based on the optimal prediction errors

$$\tilde{y}_o(t-i|t-m-1-j)\ ,\ i\ \geq 0\ ,j\ \geq 0\ .$$

The proof of the second part is based on the orthogonality principle of the optimal linear MSE estimate, see, e.g. [Söd94].

Part 1. Consider the following estimate of $u(t-m)$:

$$\hat{x}(t-m) = \gamma_{m+1}\tilde{y}_o(t|t-m-1) + \ldots + \gamma_1 \tilde{y}_o(t-m|t-m-1) \tag{B.2.1}$$

where $\tilde{y}_o(t-m-1+i|t-m-1)$, for $i = 1\ \ldots\ m+1$, are the optimal prediction errors as in Theorem B.1,

$$\tilde{y}_o(t-m-1+i|t-m-1) = y(t-m-1+i) - \hat{y}_o(t-m-1+i|t-m-1) \tag{B.2.2}$$

$$\hat{y}_o(t-m-1+i|t-m-1) = F_{y,i}(q^{-1})y(t-m-1) + F_{u,i}(q^{-1})u(t-m-1) \tag{B.2.3}$$

and $\hat{y}_o(t-m-1+i|t-m-1)$ is the i–step ahead optimal linear prediction of $y(t)$ based on the past input–output values up to time $t-m-1$. The predictor filters $F_{y,i}$ and $F_{u,i}$ are given in (B.1.9). Insert, for $i = 1 \ldots m+1$, (B.2.2)–(B.2.3) into (B.2.1).

$$\begin{aligned}\hat{x}(t-m) &= \gamma_{m+1}\left[y(t) - F_{y,m+1}(q^{-1})y(t-m-1) - F_{u,m+1}(q^{-1})u(t-m-1)\right] + \\ &+ \ldots + \gamma_1\left[y(t-m) - F_{y,1}(q^{-1})y(t-m-1) - F_{u,1}(q^{-1})u(t-m-1)\right] \quad .\end{aligned}$$

By rearranging the terms in the above expression, the estimate $\hat{x}(t-m)$ can be rewritten, as a DFE structure, with a feedback from correctly decisioned data

$$\hat{x}(t-m) = F_f(q^{-1})y(t) - F_b(q^{-1})u(t-m-1) \tag{B.2.4}$$

where

$$\begin{aligned}F_f(q^{-1}) &= \gamma_{m+1} + \gamma_m q^{-1} + \ldots + \gamma_1 q^{-m} - \\ &\quad \left[\gamma_1 F_{y,1}(q^{-1}) + \ldots + \gamma_{m+1}F_{y,m+1}(q^{-1})\right] q^{-m-1} &\text{(B.2.5)} \\ F_b(q^{-1}) &= \gamma_1 F_{u,1}(q^{-1}) + \ldots + \gamma_{m+1}F_{u,m+1}(q^{-1}) \quad . &\text{(B.2.6)}\end{aligned}$$

Consider the optimal predictors filters given in (B.1.9)

$$\begin{aligned}F_{y,i}(q^{-1}) &= \frac{M_i(q^{-1})}{M(q^{-1})} \\ F_{u,i}(q^{-1}) &= \frac{M(q^{-1})B_i(q^{-1}) - M_i(q^{-1})B(q^{-1})}{M(q^{-1})A(q^{-1})} \quad .\end{aligned}$$

With the use of the polynomial equations (B.1.5)–(B.1.8), the numerators of the optimal filters can be expressed as

$$\begin{aligned}M_i &= q^i(M - T_iN) &\text{(B.2.7)} \\ MB_i - M_iB &= q^i(T_iNB - P_iMA) \quad . &\text{(B.2.8)}\end{aligned}$$

The left hand side of (B.2.7) and (B.2.8) is a polynomial in non positive powers of q. The same will hence be true for the right hand side.

Insert, for $i = 1 \ldots m+1$, (B.2.7)–(B.2.8) into (B.2.5)–(B.2.6).

$$\begin{aligned}F_f(q^{-1}) &= \gamma_{m+1} + \gamma_m q^{-1} + \ldots + \gamma_1 q^{-m} - \\ &\quad \left[\gamma_1(M - T_1N)q + \ldots + \gamma_{m+1}(M - T_{m+1}N)q^{m+1}\right]\frac{q^{-m-1}}{M} \\ &= \frac{N}{M}\left[\gamma_1 T_1 q^{-m} + \ldots + \gamma_{m+1}T_{m+1}\right] \\ F_b(q^{-1}) &= \frac{1}{MA}\left[\gamma_1(T_1NB - P_1MA)q + \ldots + \gamma_{m+1}(T_{m+1}NB - P_{m+1}MA)q^{m+1}\right] \\ &= \frac{NBq^{m+1}}{MA}\left[\gamma_1 T_1 q^{-m} + \ldots + \gamma_{m+1}T_{m+1}\right] - \\ &\quad q\left[\gamma_1 P_1 + \ldots + \gamma_{m+1}P_{m+1}q^m\right] \quad .\end{aligned}$$

Define the following polynomials:

$$\begin{aligned}S_1(q^{-1}) &= s_0 + s_1q^{-1} + \ldots + s_mq^{-m} \triangleq \gamma_1 T_1 q^{-m} + \ldots + \gamma_{m+1}T_{m+1} &\text{(B.2.9)} \\ \tilde{L}_{1*}(q) &\triangleq \gamma_1 P_1 + \ldots + \gamma_{m+1}P_{m+1}q^m &\text{(B.2.10)} \\ L_{1*}(q) &= l_0 + l_1q + \ldots + l_mq^m \triangleq 1 - \tilde{L}_{1*}(q) \quad . &\text{(B.2.11)}\end{aligned}$$

Then, the feedforward and the feedback filters that correspond to the estimate (B.2.1) can be expressed as follows:

$$\boxed{\begin{aligned} F_f(q^{-1}) &= \frac{N}{M}S_1 \\ F_b(q^{-1}) &= q\frac{NBq^m S_1 + MA(L_{1_*}-1)}{MA} \end{aligned}} \tag{B.2.12}$$

The filters $F_f(q^{-1})$ and $F_b(q^{-1})$ are causal by definition. Compare to (B.2.5)–(B.2.6). Hence, the numerator of the feedback filter $F_b(q^{-1})$ in (B.2.12) satisfies the polynomial equation

$$q^{-1}Q = NBq^m S_1 + MA(L_{1_*}-1) \tag{B.2.13}$$

for some polynomial

$$Q(q^{-1}) = q_0 + q_1 q^{-1} + \ldots + q_{nq}q^{-nq} \tag{B.2.14}$$

$$nq = \max(nb+nn, na+nm) - 1 \; . \tag{B.2.15}$$

With the coefficients of the polynomials P_{m+1} and T_{m+1}, defined as in Theorem B.1, form the following matrices:

$$\mathbf{P} = \begin{bmatrix} p_1 & 0 & \ldots & 0 \\ p_2 & p_1 & \ddots & \\ \vdots & & \ddots & 0 \\ p_{m+1} & p_m & \ldots & p_1 \end{bmatrix} \qquad \mathbf{T} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ t_2 & 1 & \ddots & \\ \vdots & & \ddots & 0 \\ t_{m+1} & t_m & \ldots & 1 \end{bmatrix} \; . \tag{B.2.16}$$

Then, the relations (B.2.9)–(B.2.11) can be expressed in matrix form as

$$\boxed{\begin{aligned} \mathbf{s} &= \mathbf{T}\gamma \\ \mathbf{l} &= -\mathbf{P}\gamma + \mathbf{r} \end{aligned}} \tag{B.2.17}$$

where

$$\begin{aligned} \gamma &= [\; \gamma_{m+1} \;\; \ldots \;\; \gamma_1 \;]^T \\ \mathbf{s} &= [\; s_0 \;\; \ldots \;\; s_m \;]^T \\ \mathbf{l} &= [\; l_m \;\; \ldots \;\; l_0 \;]^T \\ \mathbf{r} &= [\; 0 \;\; \ldots \;\; 0 \;\; 1 \;]^T \; . \end{aligned}$$

Using (B.2.12), the estimation error is given by

$$\begin{aligned} z(t) &= u(t-m) - \hat{x}(t-m) \\ &= u(t-m) - S_1\frac{N}{M}y(t) + \left[\frac{NBq^{m+1}}{MA}S_1 - q\tilde{L}_{1_*}\right]u(t-m-1) \\ &= u(t-m) - \tilde{L}_{1_*}u(t-m) - S_1\frac{N}{M}\left[y(t) - \frac{B}{A}u(t)\right] \; . \end{aligned} \tag{B.2.18}$$

We have assumed that the system is exactly known. i.e. the data is generated by (3.2.1)

$$y(t) = \frac{B}{A}u(t) + \frac{M}{N}e(t)$$

where $e(t)$ is the innovation process in (3.2.1). Then, (B.2.18) reduces to

$$\boxed{z(t) = L_{1_*}u(t-m) - S_1 e(t)} \tag{B.2.19}$$

where S_1 and L_{1_*} are defined in (B.2.9) and (B.2.11), respectively. The estimation error can be rewritten in matrix form as

$$z(t) = \mathbf{l}^T\mathbf{u} - \mathbf{s}^T\mathbf{e} \tag{B.2.20}$$

with

$$\mathbf{u} = [\ u(t) \quad \ldots \quad u(t-m)\ ]^T \quad ; \quad \mathbf{e} = [\ e(t) \quad \ldots \quad e(t-m)\ ]^T \quad .$$

The input $u(t)$ is assumed white and $e(t)$ and $u(t)$ are assumed to be mutually uncorrelated. The MSE criterion therefore becomes

$$\begin{aligned} V(\gamma) &= \mathrm{E}z^2(t) \\ &= \mathbf{l}^T\mathbf{l} + \rho\mathbf{s}^T\mathbf{s} \\ &= (-\mathbf{P}\gamma + \mathbf{r})^T(-\mathbf{P}\gamma + \mathbf{r}) + \rho\gamma^T\mathbf{T}^T\mathbf{T}\gamma \\ &= \gamma^T(\mathbf{P}^T\mathbf{P} + \rho\mathbf{T}^T\mathbf{T})\gamma - 2\gamma^T\mathbf{P}^T\mathbf{r} + \mathbf{r}^T\mathbf{r} \end{aligned} \tag{B.2.21}$$

where (B.2.17) was used in the third equality. The matrix $(\mathbf{P}^T\mathbf{P} + \rho\mathbf{T}^T\mathbf{T})$ is positive definite. Hence, the expression (B.2.21) attains the unique global minimum for

$$\boxed{\bar{\gamma} = \left(\mathbf{P}^T\mathbf{P} + \rho\mathbf{T}^T\mathbf{T}\right)^{-1}\mathbf{P}^T\mathbf{r}} \tag{B.2.22}$$

This concludes the proof of the first part.

Part 2. Next, we shall prove that the estimate

$$\hat{u}(t-m) = \bar{\gamma}_{m+1}\tilde{y}_o(t|t-m-1) + \ldots + \bar{\gamma}_1\tilde{y}_o(t-m|t-m-1) \tag{B.2.23}$$

is, indeed, the optimal among all linear MSE estimates of $u(t-m)$ based on the optimal prediction errors

$$\tilde{y}_o(t-i|t-m-1-j)\ ,\ i\ \geq 0\ \ ,j\ \geq 0\ \ . \tag{B.2.24}$$

The proof follows by applying the orthogonality principle of the optimal linear MSE estimate. The estimate (B.2.23) will be optimal if and only if the prediction error is orthogonal to all available signals, i.e. if

$$\mathrm{E}[u(t-m) - \hat{u}(t-m)]\tilde{y}_o(t-i|t-m-1-j) = 0\ ,\ i\ \geq 0\ \ ,j\ \geq 0\ \ . \tag{B.2.25}$$

From (B.2.20) and (B.2.17), the prediction error corresponding to the estimate (B.2.23) can be rewritten in matrix form, as

$$z(t) = \left(-\bar{\gamma}^T\mathbf{P}^T + \mathbf{r}^T\right)\mathbf{u} - \bar{\gamma}^T\mathbf{T}^T\mathbf{e} \tag{B.2.26}$$

where $\bar{\gamma}$ is given by (B.2.22). From (B.1.10), the prediction error (B.2.24) is given by

$$\tilde{y}_o(t-i|t-m-1-j) = q^{-i}P_{j-i+m+1}(q^{-1})u(t) + q^{-i}T_{j-i+m+1}(q^{-1})e(t) \quad . \qquad \text{(B.2.27)}$$

Introduce the vectors $\mathbf{p}$ and $\mathbf{t}$, with $m+1$ elements

$$\begin{aligned} \mathbf{p} &= [\, 0 \;\; \dots \;\; 0 \;\; p_1 \;\; p_2 \;\; \dots \;\; p_{m+1-i} \,]^T & \text{(B.2.28)} \\ \mathbf{t} &= [\, 0 \;\; \dots \;\; 0 \;\; t_1 \;\; t_2 \;\; \dots \;\; t_{m+1-i} \,]^T & \text{(B.2.29)} \end{aligned}$$

where the zeros occupy the first i positions. If $i > m+1$ then the vectors (B.2.28), (B.2.29) contain only zeros. With the use of (B.2.28) and (B.2.29), the prediction error (B.2.27) can be rewritten as

$$\tilde{y}_o(t-i|t-m-1-j) = \mathbf{p}^T\mathbf{u} + \mathbf{t}^T\mathbf{e} + \bar{y}(t-m-2) \qquad \text{(B.2.30)}$$

where the term $\bar{y}(t-m-2)$ depends only on the past of the processes $u(t)$ and $e(t)$ up to time $t-m-2$. Apparently, the term $\bar{y}(t-m-2)$ is independent of $z(t)$ in (B.2.26). All correlations in (B.2.25) for which $i > m+1$ will be zero, since $\mathbf{p} = \mathbf{0}$ and $\mathbf{t} = \mathbf{0}$ in these cases. Assume that $i \leq m+1$. Observe that the following holds for the matrices $\mathbf{P}$ and $\mathbf{T}$ defined in (B.2.16) and the vectors $\mathbf{p}$ and $\mathbf{t}$ defined above:

$$\begin{aligned} \mathbf{p} &= \mathbf{P}\mathbf{r}_{i+1} & \text{(B.2.31)} \\ \mathbf{t} &= \mathbf{T}\mathbf{r}_{i+1} & \text{(B.2.32)} \end{aligned}$$

where

$$\mathbf{r}_{i+1} = \begin{bmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{bmatrix}^T$$

with the unit element in position $i+1$. By inserting (B.2.30), (B.2.31), (B.2.32) and (B.2.26) in (B.2.25), we obtain:

$$\begin{aligned} \mathrm{E}z(t)\tilde{y}_o(t-i|t-m-1-j) &= \left(-\bar{\gamma}^T\mathbf{P}^T + \mathbf{r}^T\right)\mathbf{p} - \rho\bar{\gamma}^T\mathbf{T}^T\mathbf{t} + \mathrm{E}z(t)\bar{y}(t-m-2) \\ &= \mathbf{r}^T\mathbf{p} - \bar{\gamma}^T\left(\mathbf{P}^T\mathbf{p} + \rho\mathbf{T}^T\mathbf{t}\right) \\ &= \mathbf{r}^T\mathbf{P}\mathbf{r}_{i+1} - \mathbf{r}^T\mathbf{P}\left(\mathbf{P}^T\mathbf{P} + \rho\mathbf{T}^T\mathbf{T}\right)^{-1}\left(\mathbf{P}^T\mathbf{P} + \rho\mathbf{T}^T\mathbf{T}\right)\mathbf{r}_{i+1} \\ &= \mathbf{r}^T\mathbf{P}\mathbf{r}_{i+1} - \mathbf{r}^T\mathbf{P}\mathbf{r}_{i+1} = 0 \end{aligned}$$

where in the third equality, the expression (B.2.22) has been used. The theorem follows ■

B.3 Proof of Theorem 3.2

The proof follows by showing that the filters (3.2.36) and (3.2.37) coincide with the optimal filters obtained from the design equations (3.2.4)–(3.2.9).

The MSE optimal DFE is given by the design equations (3.2.4)–(3.2.9). They are repeated here for the convenience of the reader:

$$\boxed{\begin{aligned} \bar{F}_f(q^{-1}) &= \frac{\bar{S}_1(q^{-1})N(q^{-1})}{M(q^{-1})} \\ \bar{F}_b(q^{-1}) &= \frac{\bar{Q}(q^{-1})}{A(q^{-1})M(q^{-1})} \end{aligned}} \tag{B.3.1}$$

where $\bar{S}_1$ and $\bar{Q}$, together with polynomials $\bar{L}_{1_*}$ and L_{2_*}, are the unique solution of the coupled polynomial equations

$$AM + q^{-1}\bar{Q} = q^m BN\bar{S}_1 + AM\bar{L}_{1_*} \tag{B.3.2}$$

$$qL_{2_*} = -\rho A_* M_* \bar{S}_1 + q^{-m} B_* N_* \bar{L}_{1_*} \tag{B.3.3}$$

with polynomial degrees

$$\begin{aligned} n\bar{s}_1 &= n\bar{l}_1 = m \\ n\bar{q} &= nl_2 = \max(na + nm, nb + nn) - 1 \quad . \end{aligned}$$

Consider the polynomial equations of the $(m+1)$–step ahead optimal linear prediction of $y(t)$, given in Theorem B.1:

$$B = P_{m+1}A + q^{-m-1}B_{m+1} \tag{B.3.4}$$

$$M = T_{m+1}N + q^{-m-1}M_{m+1} \tag{B.3.5}$$

with

$$\begin{aligned} nb_{m+1} &= \max(nb, na) - 1 \\ nm_{m+1} &= \max(nm, nn) - 1 \end{aligned}$$

and

$$P_{m+1}(q^{-1}) = p_1 + p_2 q^{-1} + \ldots + p_{m+1} q^{-m} \tag{B.3.6}$$

$$T_{m+1}(q^{-1}) = 1 + t_2 q^{-1} + \ldots + t_{m+1} q^{-m} \quad . \tag{B.3.7}$$

Insert (B.3.4)–(B.3.5) into (B.3.2)–(B.3.3). First, consider the polynomial equation (B.3.2). It results as:

$$\begin{aligned} q^{-1}Q &= q^m \left[P_{m+1}A + q^{-1-m}B_{m+1}\right] N\bar{S}_1 + A\left[T_{m+1}N + q^{-m-1}M_{m+1}\right](\bar{L}_{1_*} - 1) \\ &= \left[q^m P_{m+1}\bar{S}_1 + T_{m+1}(\bar{L}_{1_*} - 1)\right] AN + \\ &\quad + q^{-1}\left[B_{m+1}N\bar{S}_1 + AM_{m+1}q^{-m}(\bar{L}_{1_*} - 1)\right] \quad . \end{aligned} \tag{B.3.8}$$

The left hand side of (B.3.8) is a polynomial in q with only negative powers. The same is true for the term

$$q^{-1}\left[B_{m+1}N\bar{S}_1 + AM_{m+1}q^{-m}(\bar{L}_{1_*} - 1)\right] \quad .$$

Hence, equation (B.3.2) has solution if and only if the polynomials $\bar{S}_1$ and $\bar{L}_{1_*}$ satisfy

$$\boxed{\left[q^m P_{m+1}\bar{S}_1 + T_{m+1}(\bar{L}_{1_*} - 1)\right] = q^{-1}\bar{W}} \tag{B.3.9}$$

for some polynomial $\bar{W}(q^{-1})$. The same reasoning is repeated for the polynomial equation (B.3.3). It can be rewritten as

$$
\begin{aligned}
qL_{2_*} &= -\rho A_* \left(T_{(m+1)_*} N_* + q^{m+1} M_{(m+1)_*}\right) \bar{S}_1 + q^{-m} \left(P_{(m+1)_*} A_* + q^{m+1} B_{(m+1)_*}\right) N_* \bar{L}_{1_*} \\
&= \left[-\rho T_{(m+1)_*} \bar{S}_1 + q^{-m} P_{(m+1)_*} \bar{L}_{1_*}\right] A_* N_* + q \left[-\rho A_* M_{(m+1)_*} q^m \bar{S}_1 + N_* B_{(m+1)_*} \bar{L}_{1_*}\right]
\end{aligned}
$$

Then the polynomial equation has solution if and only if the polynomials $\bar{S}_1$ and $\bar{L}_{1_*}$ satisfy

$$
-\rho T_{(m+1)_*} \bar{S}_1 + q^{-m} P_{(m+1)_*} \bar{L}_{1_*} = q\bar{V}_* \tag{B.3.10}
$$

for some polynomial $\bar{V}_*(q^{-1})$. Comparing the coefficients for the positive powers of q (and the constant term) in (B.3.9) and those for the negative powers of q (and the constant term) in (B.3.10), the following system of equations is obtained for the coefficients of the polynomials $\bar{S}_1$ and $\bar{L}_{1_*}$ in the optimal DFE:

$$
\begin{aligned}
\mathbf{P}\bar{\mathbf{s}} + \mathbf{T}\bar{\mathbf{l}} &= \mathbf{r} \\
-\rho \mathbf{T}^T \bar{\mathbf{s}} + \mathbf{P}^T \bar{\mathbf{l}} &= \mathbf{0}
\end{aligned} \tag{B.3.11}
$$

where $\mathbf{P}$ and $\mathbf{T}$ are defined in (B.2.16), and

$$
\begin{aligned}
\bar{\mathbf{s}} &= [\ \bar{s}_0 \ \ \ldots \ \ \bar{s}_m\ ]^T \\
\bar{\mathbf{l}} &= [\ \bar{l}_m \ \ \ldots \ \ \bar{l}_0\ ]^T \\
\mathbf{r} &= [\ 0 \ \ \ldots \ \ 0 \ \ 1\ ]^T \ .
\end{aligned}
$$

Observe that this system of equations has a unique solution, see [StAh90].

In the proof of Theorem 3.1, it was shown that the filters (3.2.36)–(3.2.37) for the DFE have the structure

$$
\begin{aligned}
F_f(q^{-1}) &= \frac{N}{M} S_1 \\
F_b(q^{-1}) &= q\frac{NBq^m S_1 + MA(L_{1_*} - 1)}{MA}
\end{aligned} \tag{B.3.12}
$$

where the coefficients of the polynomials S_1 and L_{1_*} are obtained from the matrix equations:

$$
\begin{aligned}
\mathbf{s} &= \mathbf{T}\bar{\gamma} \\
\mathbf{l} &= -\mathbf{P}\bar{\gamma} + \mathbf{r} \\
\bar{\gamma} &= \left(\mathbf{P}^T\mathbf{P} + \rho \mathbf{T}^T\mathbf{T}\right)^{-1} \mathbf{P}^T \mathbf{r}
\end{aligned} \tag{B.3.13}
$$

See (B.2.12), (B.2.17) and (B.2.22).

In [StAh90], it is shown that the MSE optimal DFE, with filters with coprime numerator and denominator, is unique. Hence, the filters (B.3.12) will be optimal if and only if they have the same coprime factors as the filters (B.3.1). Compare the filters structure (B.3.12) to the optimal filters structure (B.3.1), using (B.3.2). Apparently:

$$\left\{ \begin{array}{c} F_f \equiv \bar{F}_f \\ F_b \equiv \bar{F}_b \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{c} S_1 \equiv \bar{S}_1 \\ L_{1_*} \equiv \bar{L}_{1_*} \end{array} \right. .$$

As will be demonstrated below, the vectors of coefficients $\mathbf{s}$ and $\mathbf{l}$ given in (B.3.13) satisfy the system of equations (B.3.11). Hence, $S_1 \equiv \bar{S}_1$ and $L_{1_*} \equiv \bar{L}_{1_*}$. Thus, the filters (3.2.36)–(3.2.37) give the MSE optimal DFE.

First, observe that, because of the Toeplitz structure of $\mathbf{P}$ and $\mathbf{T}$, the following relation holds:

$$\mathbf{PT} = \mathbf{TP} \quad . \tag{B.3.14}$$

Then, insert $\mathbf{s}$ and $\mathbf{l}$ from (B.3.13) into the left–hand side of the first of the equations (B.3.11):

$$\begin{aligned} &\mathbf{Ps} + \mathbf{Tl} = \\ &= \left[\mathbf{PT}\left(\mathbf{P}^T\mathbf{P} + \rho\mathbf{T}^T\mathbf{T}\right)^{-1}\mathbf{P}^T - \mathbf{TP}\left(\mathbf{P}^T\mathbf{P} + \rho\mathbf{T}^T\mathbf{T}\right)^{-1}\mathbf{P}^T + \mathbf{T}\right]\mathbf{r} \\ &= \left[(\mathbf{PT} - \mathbf{TP})\left(\mathbf{P}^T\mathbf{P} + \rho\mathbf{T}^T\mathbf{T}\right)^{-1}\mathbf{P}^T + \mathbf{T}\right]\mathbf{r} = \mathbf{Tr} = \mathbf{r} \end{aligned}$$

where (B.3.14) was used in the second equality. Also, insert $\mathbf{s}$ and $\mathbf{l}$ from (B.3.13) into the second equation of (B.3.11):

$$\begin{aligned} &-\rho\mathbf{T}^T\mathbf{s} + \mathbf{P}^T\mathbf{l} = \\ &= \left[-\rho\mathbf{T}^T\mathbf{T}\left(\mathbf{P}^T\mathbf{P} + \rho\mathbf{T}^T\mathbf{T}\right)^{-1}\mathbf{P}^T - \mathbf{P}^T\mathbf{P}\left(\mathbf{P}^T\mathbf{P} + \rho\mathbf{T}^T\mathbf{T}\right)^{-1}\mathbf{P}^T + \mathbf{P}^T\right]\mathbf{r} \\ &= \left[-\left(\mathbf{P}^T\mathbf{P} + \rho\mathbf{T}^T\mathbf{T}\right)\left(\mathbf{P}^T\mathbf{P} + \rho\mathbf{T}^T\mathbf{T}\right)^{-1}\mathbf{P}^T + \mathbf{P}^T\right]\mathbf{r} = \mathbf{0} \end{aligned}$$

This concludes the proof ■

B.4 Proof of Lemma 3.1

In this section, we will first prove Lemma 3.1. Then, we will show that the degradation of the optimal performance when the estimate of $u(t-m)$ is formed by using general suboptimal predictions, as in (3.3.3), is still entirely determined by the prediction errors being suboptimal.

Consider the estimate (3.3.7) of $u(t-m)$

$$\hat{u}(t-m) = \gamma_{m+1}\tilde{y}(t|t-m-1) + \ldots + \gamma_1\tilde{y}(t-m|t-m-1) \tag{B.4.1}$$

where $\tilde{y}(t-m-1+i|t-m-1)$, for $i = 1 \ \ldots \ m+1$, are the prediction errors obtained as

$$\tilde{y}(t-m-1+i|t-m-1) = y(t-m-1+i) - \hat{y}(t-m-1+i|t-m-1) \tag{B.4.2}$$

$$\hat{y}(t-m-1+i|t-m-1) = F_{y,i}(q^{-1})y(t-m-1) + F_{u,i}(q^{-1})u(t-m-1) \ . \tag{B.4.3}$$

The predictor filters $F_{y,i}(q^{-1})$, $F_{u,i}(q^{-1})$ in (B.4.3) are designed from the equations of the optimal linear prediction for the model S.

The same steps as in the proof of Theorem 3.1 can be repeated up to the expression (B.2.18) for the estimation error. The estimation error is given by

$$\begin{aligned} z(t) &= u(t-m) - \hat{u}(t-m) \\ &= u(t-m) - \tilde{L}_{1_*}u(t-m) - S_1\frac{N}{M}\left[y(t) - \frac{B}{A}u(t)\right] \end{aligned} \tag{B.4.4}$$

where S_1 and $\tilde{L}_{1_*}$ were defined in (B.2.9) and (B.2.10), respectively. The system (3.2.1) is described by the approximate model

$$y(t) = \frac{B}{A}u(t) + \frac{M}{N}\hat{e}(t)$$

where the transfer function $H = M/N$, which models the noise correlation, is assumed to be minimum phase. The model residual $\hat{e}(t)$ is then obtained as

$$\hat{e}(t) = \frac{N}{M}\left[y(t) - \frac{B}{A}u(t)\right] \ . \tag{B.4.5}$$

With use of expression (B.4.5), the estimation error can be expressed as

$$\boxed{z(t) = L_{1_*}u(t-m) - S_1\hat{e}(t)} \tag{B.4.6}$$

where $L_{1_*} = 1 - \tilde{L}_{1_*}$. The estimation error can be rewritten in matrix form as

$$z(t) = \mathbf{l}^T\mathbf{u} - \mathbf{s}^T\hat{\mathbf{e}}$$

with

$$\mathbf{u} = [\ u(t) \ \ \ldots \ \ u(t-m)\ ]^T \ \ ; \ \ \hat{\mathbf{e}} = [\ \hat{e}(t) \ \ \ldots \ \ \hat{e}(t-m)\ ]^T \ .$$

Define the following correlation matrices:

$$\mathbf{R}_{\hat{e}} = \mathrm{E}\ \hat{\mathbf{e}}\hat{\mathbf{e}}^T \tag{B.4.7}$$

$$\mathbf{R}_{\hat{e}u} = \mathrm{E}\ \hat{\mathbf{e}}\mathbf{u}^T \ . \tag{B.4.8}$$

The MSE when using the estimate (3.3.7) becomes:

$$\begin{aligned} V(\gamma) &= \mathrm{E}z^2(t) \\ &= \mathbf{l}^T\mathbf{l} + \mathbf{s}^T\mathbf{R}_{\hat{e}}\mathbf{s} - \mathbf{s}^T\mathbf{R}_{\hat{e}u}\mathbf{l} - \mathbf{l}^T\mathbf{R}_{\hat{e}u}^T\mathbf{s} \\ &= (-\mathbf{P}\gamma + \mathbf{r})^T(-\mathbf{P}\gamma + \mathbf{r}) + \gamma^T\mathbf{T}^T\mathbf{R}_{\hat{e}}\mathbf{T}\gamma \\ &\quad -\gamma^T\mathbf{T}^T\mathbf{R}_{\hat{e}u}(-\mathbf{P}\gamma + \mathbf{r}) - (-\mathbf{P}\gamma + \mathbf{r})^T\mathbf{R}_{\hat{e}u}^T\mathbf{T}\gamma \\ &= \gamma^T\left[\mathbf{P}^T\mathbf{P} + \mathbf{T}^T\mathbf{R}_{\hat{e}}\mathbf{T} + \mathbf{T}^T\mathbf{R}_{\hat{e}u}\mathbf{P} + \mathbf{P}^T\mathbf{R}_{\hat{e}u}^T\mathbf{T}\right]\gamma \\ &\quad -2\gamma^T\left[\mathbf{P}^T + \mathbf{T}^T\mathbf{R}_{\hat{e}u}\right]\mathbf{r} + \mathbf{r}^T\mathbf{r} \end{aligned} \tag{B.4.9}$$

where (B.2.17) was used in the third equality. Hence, the expression (B.4.9) attains the unique global minimum for

$$\boxed{\bar{\gamma} = \left[\mathbf{P}^T\mathbf{P} + \mathbf{T}^T\mathbf{R}_{\hat{e}}\mathbf{T} + \mathbf{T}^T\mathbf{R}_{\hat{e}u}\mathbf{P} + \mathbf{P}^T\mathbf{R}_{\hat{e}u}^T\mathbf{T}\right]^{-1}\left[\mathbf{P}^T + \mathbf{T}^T\mathbf{R}_{\hat{e}u}\right]\mathbf{r}} \tag{B.4.10}$$

The fact that the matrix

$$\left[\mathbf{P}^T\mathbf{P} + \mathbf{T}^T\mathbf{R}_{\hat{e}}\mathbf{T} + \mathbf{T}^T\mathbf{R}_{\hat{e}u}\mathbf{P} + \mathbf{P}^T\mathbf{R}_{\hat{e}u}^T\mathbf{T}\right] \tag{B.4.11}$$

is positive definite is shown below. This concludes the proof.

Next, consider the estimate (3.3.3)

$$\hat{\mathrm{u}}(t-m) = \gamma_{m+1}\hat{\hat{\mathrm{y}}}(t|t-m-1) + \ldots + \gamma_1\hat{\hat{\mathrm{y}}}(t-m|t-m-1) \tag{B.4.12}$$

obtained from the suboptimal predictions

$$\hat{\hat{\mathrm{y}}}(t-m-1+i|t-m-1) \quad . \tag{B.4.13}$$

with $i = 1, \ldots, m+1$. With the use of the optimal predictions of the output signal $y(t)$, defined in Theorem B.1, the corresponding prediction errors, for $i = 1, \ldots, m+1$, can be rewritten as

$$\begin{aligned}
\tilde{\hat{\mathrm{y}}}(t-m-1+i|t-m-1) &= y(t-m-1+i) - \hat{\hat{\mathrm{y}}}(t-m-1+i|t-m-1) \\
&= [y(t-m-1+i) - \hat{y}_o(t-m-1+i|t-m-1)] + \\
&\quad + \left[\hat{y}_o(t-m-1+i|t-m-1) - \hat{\hat{\mathrm{y}}}(t-m-1+i|t-m-1)\right] \\
&= \tilde{y}_o(t-m-1+i|t-m-1) + \epsilon_i(\hat{\hat{\mathrm{y}}}, t-m-1)
\end{aligned} \tag{B.4.14}$$

where $\tilde{y}_o(t-m-1+i|t-m-1)$ are the optimal prediction errors, and the terms $\epsilon_i(\hat{\hat{\mathrm{y}}}, t-m-1)$ depend on the predictors (B.4.13) and on past values of $u(t)$ and $e(t)$ up to $t-m-1$. The optimal prediction errors depend on past values of $u(t)$ and $e(t)$ more recent that $t-m$. Apparently, the two terms in (B.4.14) are mutually uncorrelated. Introduce the following vectors:

$$\begin{aligned}
\tilde{\mathbf{y}}_o &= [\ \tilde{y}_o(t|t-m-1) \ \ \ldots \ \ \tilde{y}_o(t-m|t-m-1)\ ]^T \\
\epsilon &= [\ \epsilon_{m+1}(\hat{\hat{\mathrm{y}}}, t-m-1) \ \ \ldots \ \ \epsilon_1(\hat{\hat{\mathrm{y}}}, t-m-1)\ ]^T \quad .
\end{aligned}$$

Then, the estimate (B.4.12) can be rewritten in matrix form as

$$\hat{\mathrm{u}}(t-m) = \gamma^T\left(\tilde{\mathbf{y}}_o + \epsilon\right) \quad . \tag{B.4.15}$$

Furthermore, define the following correlation matrices:

$$\mathbf{R}_{\tilde{\mathbf{y}}_o} = \mathrm{E}\ \tilde{\mathbf{y}}_o\tilde{\mathbf{y}}_o^T \tag{B.4.16}$$

$$\mathbf{R}_\epsilon = \mathrm{E}\ \epsilon\epsilon^T \tag{B.4.17}$$

$$\mathbf{r}_{u\tilde{\mathbf{y}}_o} = \mathrm{E}\ u(t-m)\tilde{\mathbf{y}}_o \quad . \tag{B.4.18}$$

With the use of (B.4.16)–(B.4.18) and (B.4.15), the MSE criterion becomes

$$\begin{aligned} V(\gamma, \hat{\hat{\mathrm{y}}}) &= \mathrm{E}\ (u(t-m) - \hat{\hat{\mathrm{u}}}(t-m))^2 \\ &= 1 + \gamma^T \left[\mathbf{R}_{\tilde{\mathbf{y}}_o} + \mathbf{R}_\epsilon\right] \gamma - 2\gamma^T \mathbf{r}_{u\tilde{\mathbf{y}}_o} \\ &= V(\gamma, \hat{y}_o) + \gamma^T \mathbf{R}_\epsilon \gamma \end{aligned} \tag{B.4.19}$$

where $V(\gamma, \hat{y}_o)$ is the criterion that would result if the optimal predictions were used in the estimate (B.4.12). Hence, the performance degradation that results otherwise is given by the term

$$\gamma^T \mathbf{R}_\epsilon \gamma$$

that is entirely determined by the choice of the suboptimal predictions (B.4.13).

Finally, observe that the matrix (B.4.11) corresponds to the matrix

$$\left[\mathbf{R}_{\tilde{\mathbf{y}}_o} + \mathbf{R}_\epsilon\right] \quad .$$

It is positive definite except for degenerate cases ■

B.5 Proof of Theorem 3.3

Consider the multiple prediction method. In the case of pure filtering ($m = 0$), the estimate of $u(t)$ is formed as

$$\hat{u}(t) = \gamma \tilde{y}(t|t-1) \tag{B.5.1}$$

where the prediction error is obtained as

$$\tilde{y}(t|t-1) = y(t) - \hat{y}(t|t-1) \quad . \tag{B.5.2}$$

The variance of the prediction error is

$$W(S) = \mathrm{E}\tilde{y}^2(t|t-1) \quad . \tag{B.5.3}$$

The notation $W(S)$ in (B.5.3) stresses the dependence on the model S from which the optimal (for the model) predictor in (B.5.2) is calculated. With the use of the one step ahead optimal linear prediction of $y(t)$, denoted by $\hat{y}_o(t|t-1)$, see Theorem B.1, the prediction error (B.5.2) can be rewritten as

$$\begin{aligned} \tilde{y}(t|t-1) &= [y(t) - \hat{y}_o(t|t-1)] + [\hat{y}_o(t|t-1) - \hat{y}(t|t-1)] \\ &= g_0 u(t) + e(t) + \epsilon_1(S, t-1) \end{aligned} \tag{B.5.4}$$

where g_0 is the first impulse response coefficient of the system S_T, and $\epsilon_1(S, t-1)$ depends on the model S and on past values of $u(t)$ and $e(t)$ up to time $t-1$. The term $\epsilon_1(S, t-1)$ is mutually uncorrelated with $u(t)$, hence

$$\mathrm{E}\ u(t)\tilde{y}(t|t-1) = g_0 \quad . \tag{B.5.5}$$

The MSE criterion is given by

$$\begin{aligned} V(S, \gamma) &= \mathrm{E}\ z^2(t) = [u(t) - \gamma \tilde{y}(t|t-1)] \\ &= 1 + \gamma^2 W(S) - 2\gamma g_0 \end{aligned} \tag{B.5.6}$$

where (B.5.5) was utilized. The suboptimal estimate of $u(t)$ is then calculated as

$$\hat{u}(t) = \bar{\gamma}\tilde{y}(t|t-1) \tag{B.5.7}$$

with the scalar gain $\bar{\gamma}$ obtained by minimizing the MSE criterion (B.5.6). The corresponding filtering error variance for $u(t)$ is found as

$$V(S,\bar{\gamma}) = \frac{W(S) - g_0^2}{W(S)} \quad . \tag{B.5.8}$$

The estimation performance depends only on the model S. Consider the derivative of $V(S,\bar{\gamma})$ with respect to the quantity $W(S)$:

$$\frac{\delta V(S,\bar{\gamma})}{\delta W(S)} = \frac{g_0^2}{W^2(S)} > 0$$

Since the first derivative is positive, the estimation performance is minimized by minimizing $W(S)$ over the model class.

Now consider the nominal design. The matrices $\mathbf{P}$, $\mathbf{T}$ correspond to the scalars

$$\begin{aligned} \mathbf{P} &= b_0 \\ \mathbf{T} &= 1 \end{aligned}$$

where b_0 is the first impulse response coefficient of the model S. The scalar gain $\bar{\gamma}$ given in Theorem 3.1 is then found as

$$\bar{\gamma} = \frac{b_0}{b_0^2 + \hat{\rho}} \tag{B.5.9}$$

where $\hat{\rho}$ is the variance of the model residual[3]

$$\hat{e}(t) = \frac{N}{M}\left[y(t) - \frac{B}{A}u(t)\right] \quad . \tag{B.5.10}$$

Then, from expressions (3.3.19), (3.3.20):

$$\begin{aligned} s_0 &= \frac{b_0}{b_0^2 + \hat{\rho}} \\ l_0 &= -\frac{b_0^2}{b_0^2 + \hat{\rho}} + 1 \end{aligned}$$

Apparently, from (B.5.10):

$$\mathrm{E}\ \hat{e}(t)u(t) = g_0 - b_0 \tag{B.5.11}$$

where g_0 is the first impulse response coefficient of the channel $G_T(q^{-1})$ in (3.2.1). From the expression (B.4.6), the estimation error is given by

$$\begin{aligned} z(t) &= u(t) - \hat{u}(t) = l_0 u(t) - s_0\hat{e}(t) \\ &= \frac{\hat{\rho}}{b_0^2 + \hat{\rho}}u(t) - \frac{b_0}{b_0^2 + \hat{\rho}}\hat{e}(t) \quad . \end{aligned}$$

[3]The transfer function $H = M/N$ that models the noise correlation is assumed to be minimum phase.

Hence, the MSE criterion results as

$$
\begin{aligned}
V(S) &= \mathrm{E}\ z^2(t) = \frac{\hat{\rho}^2}{[b_0^2+\hat{\rho}]^2} + \frac{b_0^2}{[b_0^2+\hat{\rho}]^2}\hat{\rho} - 2\frac{b_0\hat{\rho}}{[b_0^2+\hat{\rho}]^2}(g_0 - b_0) \\
&= \hat{\rho}\frac{3b_0^2+\hat{\rho}(1-2g_0b_0)}{[b_0^2+\hat{\rho}]^2} \qquad \text{(B.5.12)}
\end{aligned}
$$

where the use of (B.5.11) was made in the first equality. Expression (B.5.12) depends nonlinearly on the model via the parameters b_0 and $\hat{\rho}$. It is not suitable for inferring strategies for the optimization of the modeling stage ■

B.6 Derivation of Expressions Used in Example 3.1

In order to simplify the mathematical expressions appearing in the example, the following notation will be used:

$$
\begin{aligned}
\mathbf{c} &\triangleq [\ c_0 \quad c_1\ ]^T \ ; \ \mathbf{d} \triangleq [\ d_1 \quad d_0\ ]^T \\
\mathbf{s} &\triangleq [\ s_0 \quad s_1\ ]^T \ ; \ \mathbf{l} \triangleq [\ l_1 \quad l_0\ ]^T \\
\gamma &\triangleq [\ \gamma_2 \quad \gamma_1\ ]^T \ ; \ \mathbf{g} \triangleq [\ g_1 \quad g_0\ ]^T \\
\mathbf{G}_1 &\triangleq \begin{bmatrix} 0 & g_2 \\ g_2 & g_1 \end{bmatrix}^T \ ; \ \mathbf{G}_2 \triangleq \begin{bmatrix} g_3 & 0 \\ 0 & 0 \end{bmatrix}^T \\
\mathbf{A}_{dir} &\triangleq \begin{bmatrix} \rho_0 + g_0^2 + g_1^2 & \rho_1 + g_0g_1 \\ \rho_1 + g_0g_1 & \rho_0 + g_0^2 + g_3^2 \end{bmatrix} \\
\mathbf{A}_n &\triangleq \begin{bmatrix} \rho_0 + g_0^2 + g_1^2 + g_3^2 & g_0g_1 \\ g_0g_1 & \rho_0 + g_0^2 + g_3^2 \end{bmatrix} .
\end{aligned}
$$

Direct Method
Consider the estimation error (3.5.4)

$$
z(t) = u(t-1) - F_f(q^{-1})y(t) + F_b(q^{-1})u(t-2) \ . \qquad \text{(B.6.1)}
$$

From (3.5.1), (3.5.3) and (B.6.1), we obtain

$$
\begin{aligned}
z(t) &= \left[q^{-1} - \left(c_0 + c_1q^{-1}\right)\left(g_0 + g_1q^{-1} + g_2q^{-2} + g_3q^{-3}\right) + \left(d_0 + d_1q^{-1}\right)q^{-2}\right]u(t) \\
&\quad - \left(c_0 + c_1q^{-1}\right)v(t) \\
&= \left[-c_0g_0 + (1 - c_0g_1 - c_1g_0)\,q^{-1} + (-c_0g_2 - c_1g_1 + d_0)\,q^{-2}\right. \\
&\quad \left. + (-c_0g_3 - c_1g_2 + d_1)\,q^{-3} - c_1g_3q^{-4}\right]u(t) - \left(c_0 + c_1q^{-1}\right)v(t) \ . \qquad \text{(B.6.2)}
\end{aligned}
$$

The contributions of $u(t-2)$ and $u(t-3)$ to the variance of $z(t)$ can be eliminated by the choice

$$
\begin{aligned}
d_0 &= c_0g_2 + c_1g_1 \\
d_1 &= c_0g_3 + c_1g_2
\end{aligned}
$$

which is (3.5.6):

$$\hat{\mathbf{d}} = [\mathbf{G}_1 + \mathbf{G}_2]\mathbf{c} \quad .$$

The resulting MSE criterion is obtained by substituting (B.6.2) into (3.5.5):

$$\begin{aligned} V(c_0, c_1) &= \mathrm{E}z^2(t) \\ &= (c_0 g_0)^2 + (1 - c_0 g_1 - c_1 g_0)^2 + (c_1 g_3)^2 + \rho_0 (c_0^2 + c_1^2) + 2\rho_1 c_0 c_1 \\ &= 1 + c_0^2(g_0^2 + g_1^2 + \rho_0) + c_1^2(g_0^2 + g_3^2 + \rho_0) + 2c_0c_1(g_0g_1 + \rho_1) - 2c_0g_1 - 2c_1g_0 \end{aligned}$$

which is (3.5.7) with the use of the notation above:

$$V(\mathbf{c}) = 1 + \mathbf{c}^T \mathbf{A}_{dir} \mathbf{c} - 2\mathbf{c}^T \mathbf{g} \quad .$$

Indirect Methods
Consider the model residual

$$\begin{aligned} \hat{e}(t) &= y(t) - B(q^{-1})u(t) \\ &= \Big[(g_0 - b_0) + (g_1 - b_1)q^{-1} + (g_2 - b_2)q^{-2} + g_3 q^{-3}\Big] u(t) + v(t) \quad . \end{aligned}$$

The contributions of $u(t)$, $u(t-1)$ and $u(t-2)$ to the variance of $\hat{e}(t)$ can be eliminated by the choice

$$\begin{aligned} \hat{B}(q^{-1}) &= g_0 + g_1 q^{-1} + g_2 q^{-2} \\ \hat{\rho} &= \rho_0 + g_3^2 \end{aligned}$$

which is (3.5.9). By solving (3.2.23), (3.2.24) in Theorem 3.1 for the model $\hat{B}(q^{-1})$,

$$\begin{aligned} B &= P_2 + q^{-2} B_2 \\ 1 &= T_2 1 + q^{-2} M_2 \end{aligned}$$

the matrices $\mathbf{P}$ and $\mathbf{T}$ are found as

$$\mathbf{P} = \begin{bmatrix} g_0 & 0 \\ g_1 & g_0 \end{bmatrix} \qquad \mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad .$$

The vectors of coefficients $\mathbf{s}$ and $\mathbf{l}$ for the polynomials S_1 and L_{1_*} are calculated from the expressions (3.3.18):

$$\begin{aligned} \mathbf{s} &= \mathbf{T}\gamma \\ \mathbf{l} &= -\mathbf{P}\gamma + \mathbf{r} \end{aligned}$$

with $\mathbf{r} = [0\ 1]^T$. Then, the polynomials S_1 and L_{1_*} are obtained as

$$\begin{aligned} S_1 &= \gamma_2 + \gamma_1 q^{-1} \\ L_{1_*} &= (-\gamma_1 g_0 - \gamma_2 g_1 + 1) - \gamma_2 g_0 q \quad . \end{aligned}$$

The polynomial Q in the feedback filter in (3.3.15) is then obtained form (3.3.15) as

$$\begin{aligned} q^{-1} Q(q^{-1}) &= q\hat{B} S_1 + L_{1_*} - 1 \\ &= q(g_0 + g_1 q^{-1} + g_2 q^{-2})(\gamma_2 + \gamma_1 q^{-1}) - (\gamma_1 g_0 + \gamma_2 g_1) - \gamma_2 g_0 q \\ &= q^{-1}(g_1\gamma_1 + g_2\gamma_2 + g_2\gamma_1 q^{-1}) \quad . \end{aligned}$$

The feedback filter is given by

$$F_b(q^{-1}) = Q(q^{-1}) \quad .$$

Hence:

$$\begin{aligned} d_0 &= g_1 c_1 + g_2 c_0 \\ d_1 &= g_2 c_1 \end{aligned}$$

which is (3.5.11). From (B.6.2), it is seen that also the unmodeled term

$$-c_0 g_3 u(t-3)$$

will give a contribution to the MSE criterion. Hence, expression (3.5.13) is found.

$$V(\mathbf{c}) = 1 + \mathbf{c}^T \mathbf{A}_{dir} \mathbf{c} - 2\mathbf{c}^T \mathbf{g} + \mathbf{c}^T \mathbf{G}_2^2 \mathbf{c} \quad .$$

Finally, for the nominal design the value of the vector $\bar{\gamma}$ is found as

$$\bar{\gamma} = [\mathbf{P}^T\mathbf{P} + \hat{\rho}\mathbf{T}^T\mathbf{T}]^{-1}\mathbf{P}^T\mathbf{r}$$

with

$$\mathbf{P}^T\mathbf{r} = [g_1 \; g_0]^T = \mathbf{g}$$

and

$$[\mathbf{P}^T\mathbf{P} + \hat{\rho}\mathbf{T}^T\mathbf{T}] = \begin{bmatrix} g_0^2 + g_1^2 + \hat{\rho} & g_0 g_1 \\ g_0 g_1 & g_0^2 + \hat{\rho} \end{bmatrix} = \mathbf{A}_n$$

and the expressions (3.5.15) are found ■

B.7 Derivation of Expressions Used in Example 3.2

In order to simplify the mathematical expressions appearing in the example, the following notation will be used:

$$\begin{aligned} \mathbf{c} &\triangleq [\, c_0 \quad c_1 \quad c2 \,]^T \;\; ; \;\; \mathbf{d} \triangleq [\, d_2 \quad d_1 \quad d_0 \,]^T \\ \gamma &\triangleq [\, \gamma_2 \quad \gamma_1 \,]^T \\ \mathbf{g} &\triangleq [\, g_1 \quad g_0 \,]^T \;\; ; \;\; \mathbf{g}_1 \triangleq [\, \mathbf{g} \quad 0 \,]^T \\ \mathbf{G} &\triangleq \begin{bmatrix} 0 & g_3 & g_2 \\ g_3 & g_2 & g_1 \\ g_2 & g_1 & g_0 \end{bmatrix}^T \\ \mathbf{A}_{dir} &\triangleq \begin{bmatrix} \rho_0 + g_0^2 + g_1^2 & \rho_1 + g_0 g_1 & \rho_2 \\ \rho_1 + g_0 g_1 & \rho_0 + g_0^2 & \rho_1 \\ \rho_2 & \rho_1 & \rho_0 + g_3^2 \end{bmatrix} \quad . \end{aligned}$$

Direct Method
Consider the estimation error (3.5.4)

$$z(t) = u(t-1) - F_f(q^{-1})y(t) + F_b(q^{-1})u(t-2) \quad . \qquad \text{(B.7.1)}$$

From (3.5.16), (3.5.17) and (B.7.1), we obtain

$$\begin{aligned} z(t) &= \Big[q^{-1} - \left(c_0 + c_1 q^{-1} + c_2 q^{-2}\right)\left(g_0 + g_1 q^{-1} + g_2 q^{-2} + g_3 q^{-3}\right) + \\ &\quad + \left(d_0 + d_1 q^{-1} + d_2 q^{-2}\right) q^{-2}\Big] u(t) - \left(c_0 + c_1 q^{-1} + c_2 q^{-2}\right) v(t) \\ &= \Big[-c_0 g_0 + (1 - c_0 g_1 - c_1 g_0)\, q^{-1} + (-c_0 g_2 - c_1 g_1 - c_2 g_0 + d_0)\, q^{-2} + \\ &\quad + (-c_0 g_3 - c_1 g_2 - c_2 g_1 + d_1)\, q^{-3} + (-c_1 g_3 - c_2 g_2 + d_2)\, q^{-4} - \\ &\quad - c_2 g_3 q^{-5}\Big] u(t) - \left(c_0 + c_1 q^{-1} + c_2 q^{-2}\right) v(t) \quad . \end{aligned} \tag{B.7.2}$$

The contributions of $u(t-2)$, $u(t-3)$ and $u(t-4)$ to the variance of $z(t)$ can be eliminated by the choice

$$\begin{aligned} d_2 &= c_1 g_3 + c_2 g_2 \\ d_1 &= c_0 g_3 + c_1 g_2 + c_2 g_1 \\ d_0 &= c_0 g_2 + c_1 g_1 + c_2 g_0 \end{aligned}$$

which is (3.5.18):

$$\hat{\mathbf{d}} = \mathbf{G}\mathbf{c} \quad .$$

The resulting MSE criterion is obtained by substituting (B.7.2) into (3.5.5):

$$\begin{aligned} V(c_0, c_1, c_2) &= \mathrm{E} z^2(t) \\ &= (c_0 g_0)^2 + (1 - c_0 g_1 - c_1 g_0)^2 + (c_2 g_3)^2 + \rho_0 (c_0^2 + c_1^2 + c_2^2) + \\ &\quad + 2\rho_1 (c_0 c_1 + c_1 c_2) + 2\rho_2 c_0 c_2 \\ &= 1 + c_0^2 (g_0^2 + g_1^2 + \rho_0) + c_1^2 (g_0^2 + \rho_0) + c_2^2 (\rho_0 + g_3^2) + \\ &\quad + 2 c_0 c_1 (g_0 g_1 + \rho_1) + 2\rho_1 c_1 c_2 + 2\rho_2 c_0 c_2 - 2 c_0 g_1 - 2 c_1 g_0 \end{aligned}$$

which is (3.5.19), by the use of the notation introduced above:

$$V(\mathbf{c}) = 1 + \mathbf{c}^T \mathbf{A}_{dir} \mathbf{c} - 2\mathbf{c}^T \mathbf{g}_1 \quad .$$

Multiple Prediction Method

Consider the polynomial equations for optimal prediction. See Theorem B.1.

$$B = P_i + q^{-i} B_i \tag{B.7.3}$$

$$1 = T_i N + q^{-i} M_i \tag{B.7.4}$$

where

$$\begin{aligned} nb_i &= nb - 1 \\ nm_i &= nn - 1 \end{aligned}$$

and

$$P_i(q^{-1}) = p_1 + \ldots + p_i q^{-i+1} \tag{B.7.5}$$

$$T_i(q^{-1}) = 1 + \ldots + t_i q^{-i+1} \quad . \tag{B.7.6}$$

For the model class $\mathcal{M}_2$, the equations (B.7.3)–(B.7.6) are solved for

$$B_1 = b_1 + b_2 q^{-1} \tag{B.7.7}$$

$$B_2 = b_2 \tag{B.7.8}$$

$$M_1 = -n_1 \tag{B.7.9}$$

$$M_2 = n_1^2 \quad . \tag{B.7.10}$$

From expressions (B.1.9) in Theorem B.1, the optimal predictor filters are given as

$$\begin{aligned} F_{y,i}(q^{-1}) &= M_i(q^{-1}) && \text{(B.7.11)} \\ F_{u,i}(q^{-1}) &= B_i(q^{-1}) - M_i(q^{-1})B(q^{-1}) \ . && \text{(B.7.12)} \end{aligned}$$

By inserting (B.7.7)–(B.7.10) into (B.7.11)–(B.7.12), the filters structures in (3.5.24)–(3.5.25)

$$\begin{aligned} F_{y,i}(q^{-1}) &= m && \text{(B.7.13)} \\ F_{u,i}(q^{-1}) &= r_0 + r_1 q^{-1} + r_2 q^{-2} && \text{(B.7.14)} \end{aligned}$$

are found, where $i = 1, 2$. Consider the one step prediction error

$$\tilde{y}(t-1|t-2) = y(t-1) - F_{y,1}y(t-2) - F_{u,1}u(t-2) \quad \text{(B.7.15)}$$

From (3.5.1), (3.5.3) and (B.7.13)–(B.7.14), we obtain

$$\begin{aligned} \tilde{y}(t-1|t-2) &= \Big[\left(q^{-1} - mq^{-2}\right)\left(g_0 + g_1 q^{-1} + g_2 q^{-2} + g_3 q^{-3}\right) \\ &\quad - \left(r_0 + r_1 q^{-1} + r_2 q^{-2}\right) q^{-2}\Big] u(t) + \left(q^{-1} - mq^{-2}\right) v(t) \\ &= \Big[g_0 q^{-1} + (g_1 - mg_0 - r_0)\, q^{-2} + (g_2 - mg_1 - r_1)\, q^{-3} + \\ &\quad + (g_3 - mg_2 - r_2)\, q^{-4} - mg_3 q^{-5}\Big] u(t) + \left(q^{-1} - mq^{-2}\right) v(t) \ . \quad \text{(B.7.16)} \end{aligned}$$

The contributions of $u(t-2)$, $u(t-3)$ and $u(t-4)$ to the variance of $\tilde{y}(t-1|t-2)$ can be eliminated by the choice

$$\begin{aligned} r_2 &= g_3 - mg_2 \\ r_1 &= g_2 - mg_1 \\ r_0 &= g_1 - mg_0 \end{aligned}$$

which corresponds to (3.5.28). The coefficient $\hat{m}_1$ is found by minimizing the variance of the prediction error, obtained by substituting (B.7.16) into (3.5.23):

$$\begin{aligned} W_1(m) &= \mathrm{E}\tilde{y}(t-1|t-2)^2 \\ &= g_0^2 + m^2 g_3^2 + (1+m^2)\rho_0 - 2m\rho_1 && \text{(B.7.17)} \end{aligned}$$

Expression (B.7.17) is minimized by the value of m given as

$$\hat{m}_1 = \frac{\rho_1}{\rho_0 + g_3^2}$$

which is (3.5.26). Consider the two step prediction error

$$\tilde{y}(t|t-2) = y(t) - F_{y,2}y(t-2) - F_{u,2}u(t-2) \quad \text{(B.7.18)}$$

From (3.5.1), (3.5.3) and (B.7.13)–(B.7.14), we obtain

$$\begin{aligned} \tilde{y}(t|t-2) &= \Big[\left(1 - mq^{-2}\right)\left(g_0 + g_1 q^{-1} + g_2 q^{-2} + g_3 q^{-3}\right) - \\ &\quad - \left(r_0 + r_1 q^{-1} + r_2 q^{-2}\right) q^{-2}\Big] u(t) + \left(1 - mq^{-2}\right) v(t) \\ &= \Big[g_0 + g_1 q^{-1} + (g_2 - mg_0 - r_0)\, q^{-2} + (g_3 - mg_1 - r_1)\, q^{-3} + \\ &\quad + (-mg_2 - r_2)\, q^{-4} - mg_3 q^{-5}\Big] u(t) + \left(q^{-1} - mq^{-2}\right) v(t) \ . \quad \text{(B.7.19)} \end{aligned}$$

The contributions of $u(t-2)$, $u(t-3)$ and $u(t-4)$ to the variance of $\tilde{y}(t|t-2)$ can be eliminated by the choice

$$\begin{aligned} r_2 &= -mg_2 \\ r_1 &= g_3 - mg_1 \\ r_0 &= g_2 - mg_0 \end{aligned}$$

which corresponds to (3.5.29). The coefficient $\hat{m}_2$ is found by minimizing the variance of the prediction error, obtained by substituting (B.7.19) into (3.5.23):

$$\begin{aligned} W_2(m) &= \mathrm{E}\tilde{y}(t|t-2)^2(t) \\ &= g_0^2 + g_1^2 + m^2 g_3^2 + (1+m^2)\rho_0 - 2m\rho_2 \quad . \end{aligned} \tag{B.7.20}$$

Expression (B.7.20) is minimized by the value of m given as

$$\hat{m}_2 = \frac{\rho_2}{\rho_0 + g_3^2}$$

which is (3.5.27). From expressions (3.2.36)–(3.2.37) in Theorem 3.2, see Figure 3.5, the feedforward and the feedback filters will be obtained as

$$\begin{aligned} F_f(q^{-1}) &= c_0 + c_1 q^{-1} + c_2 q^{-2} \\ &= \gamma_2 + \gamma_1 q^{-1} - \left[\gamma_1 F_{y,1}(q^{-1}) + \gamma_2 F_{y,2}(q^{-1})\right] q^{-2} \end{aligned} \tag{B.7.21}$$

$$\begin{aligned} F_b(q^{-1}) &= d_0 + d_1 q^{-1} + d_2 q^{-2} \\ &= \gamma_1 F_{u,1}(q^{-1}) + \gamma_2 F_{u,2}(q^{-1}) \quad . \end{aligned} \tag{B.7.22}$$

From (3.5.26)–(3.5.29), and (B.7.21)–(B.7.22),

$$\begin{aligned} c_0 &= \gamma_2 \\ c_1 &= \gamma_1 \\ c_2 &= -\gamma_1 \hat{m}_1 - \gamma_2 \hat{m}_2 \end{aligned}$$

which corresponds to (3.5.33), with

$$\mathbf{B}_p \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\hat{m}_2 & -\hat{m}_1 \end{bmatrix} \quad .$$

For the feedback filter, from the same expressions as above,

$$\begin{aligned} F_b(q^{-1}) = {} & \gamma_1(g_1 - \hat{m}_1 g_0) + \gamma_2(g_2 - \hat{m}_2 g_0) + [\gamma_1(g_2 - \hat{m}_1 g_1) + \gamma_2(g_3 - \hat{m}_2 g_1)]q^{-1} + \\ & + [\gamma_1(g_3 - \hat{m}_1 g_2) - \gamma_2 \hat{m}_2 g_2]q^{-2} \\ = {} & g_2\gamma_2 + g_1\gamma_1 - g_0(\gamma_1\hat{m}_1 + \gamma_2\hat{m}_2) + [g_3\gamma_2 + g_2\gamma_1 - g_1(\gamma_1\hat{m}_1 + \gamma_2\hat{m}_2)]q^{-1} + \\ & + g_3\gamma_1 - g_2(\gamma_1\hat{m}_1 + \gamma_2\hat{m}_2)]q^{-2} \quad . \end{aligned}$$

By expressing the coefficients of F_b in matrix form, and using (3.5.33), expression (3.5.34) is then found.

Next, we shall prove that the performance obtained by the mp–method given in (3.5.36) coincides with that obtained by the dir–method given in (3.5.20).

$$V_{dir} = 1 - \mathbf{g}_1^T \mathbf{A}_{dir}^{-1} \mathbf{g}_1 \tag{B.7.23}$$

$$V_p = 1 - \mathbf{g}^T [\mathbf{B}_p^T \mathbf{A}_{dir} \mathbf{B}_p]^{-1} \mathbf{g} \quad . \tag{B.7.24}$$

Comparing $\mathbf{g}_1$ to $\mathbf{g}$, the two performances V_p and V_{dir} will coincide if and only if

$$[\mathbf{A}_{dir}^{-1}]_{1,2} = [\mathbf{B}_p^T \mathbf{A}_{dir} \mathbf{B}_p]^{-1}$$

where the left hand side of the equality represent the subblock of

$$\mathbf{A}_{dir}^{-1}$$

formed with the first two rows and columns.

Let us compute

$$[\mathbf{B}_p^T \mathbf{A}_{dir} \mathbf{B}_p]^{-1} \quad .$$

With the use of the notation introduced above,

$$\begin{aligned}
\mathbf{B}_p^T \mathbf{A}_{dir} &= \begin{bmatrix} 1 & 0 & -\hat{m}_2 \\ 0 & 1 & -\hat{m}_1 \end{bmatrix} \begin{bmatrix} \rho_0 + g_0^2 + g_1^2 & \rho_1 + g_0 g_1 & \rho_2 \\ \rho_1 + g_0 g_1 & \rho_0 + g_0^2 & \rho_1 \\ \rho_2 & \rho_1 & \rho_0 + g_3^2 \end{bmatrix} \\
&= \begin{bmatrix} \rho_0 + g_0^2 + g_1^2 - \rho_2 \hat{m}_2 & \rho_1 + g_0 g_1 - \rho_1 \hat{m}_2 & \rho_2 - (\rho_0 + g_3^2)\hat{m}_2 \\ \rho_1 + g_0 g_1 - \rho_2 \hat{m}_1 & \rho_0 + g_0^2 - \rho_1 \hat{m}_1 & \rho_1 - (\rho_0 + g_3^2)\hat{m}_1 \end{bmatrix} \\
&= \frac{1}{\rho_0 + g_3^2} \begin{bmatrix} (\rho_0 + g_0^2 + g_1^2)(\rho_0 + g_3^2) - \rho_2^2 & (\rho_1 + g_0 g_1)(\rho_0 + g_3^2) - \rho_1 \rho_2 & 0 \\ (\rho_1 + g_0 g_1)(\rho_0 + g_3^2) - \rho_1 \rho_2 & (\rho_0 + g_0^2)(\rho_0 + g_3^2) - \rho_1^2 & 0 \end{bmatrix}
\end{aligned}$$

where in the last equality we have used the expressions (3.5.26), (3.5.27) for $\hat{m}_1$ and $\hat{m}_2$. Hence,

$$\mathbf{B}_p^T \mathbf{A}_{dir} \mathbf{B}_p = \frac{1}{\rho_0 + g_3^2} \begin{bmatrix} (\rho_0 + g_0^2 + g_1^2)(\rho_0 + g_3^2) - \rho_2^2 & (\rho_1 + g_0 g_1)(\rho_0 + g_3^2) - \rho_1 \rho_2 \\ (\rho_1 + g_0 g_1)(\rho_0 + g_3^2) - \rho_1 \rho_2 & (\rho_0 + g_0^2)(\rho_0 + g_3^2) - \rho_1^2 \end{bmatrix} . \tag{B.7.25}$$

The determinant of the matrix in (B.7.25) is given by

$$\begin{aligned}
\Delta_1 &= \mathrm{Det}((\rho_0 + g_3^2)\mathbf{B}_p^T \mathbf{A}_{dir} \mathbf{B}_p) \\
&= \Big\{ \Big[(\rho_0 + g_0^2 + g_1^2)(\rho_0 + g_3^2) - \rho_2^2\Big] \Big[(\rho_0 + g_0^2)(\rho_0 + g_3^2) - \rho_1^2\Big] - \\
&\quad - \Big[(\rho_1 + g_0 g_1)(\rho_0 + g_3^2) - \rho_1 \rho_2\Big]^2 \Big\}
\end{aligned} \tag{B.7.26}$$

Hence,

$$\begin{aligned}
&[\mathbf{B}_p^T \mathbf{A}_{dir} \mathbf{B}_p]^{-1} = \\
&= \frac{\rho_0 + g_3^2}{\Delta_1} \begin{bmatrix} (\rho_0 + g_0^2)(\rho_0 + g_3^2) - \rho_1^2 & -(\rho_1 + g_0 g_1)(\rho_0 + g_3^2) + \rho_1 \rho_2 \\ -(\rho_1 + g_0 g_1)(\rho_0 + g_3^2) + \rho_1 \rho_2 & (\rho_0 + g_0^2 + g_1^2)(\rho_0 + g_3^2) - \rho_2^2 \end{bmatrix}
\end{aligned} \tag{B.7.27}$$

Next, let us calculate the determinant of the matrix $\mathbf{A}_{dir}$:

$$\begin{aligned}
\Delta_2 &= \mathrm{Det}(\mathbf{A}_{dir}) \\
&= (\rho_0 + g_0^2 + g_1^2) \Big[(\rho_0 + g_0^2)(\rho_0 + g_3^2) - \rho_1^2\Big] - \\
&\quad -(\rho_1 + g_0 g_1) \Big[(\rho_1 + g_0 g_1)(\rho_0 + g_3^2) - \rho_1 \rho_2\Big] + \\
&\quad +\rho_2 \Big[(\rho_1 + g_0 g_1)\rho_1 - \rho_2(\rho_0 + g_0^2)\Big]
\end{aligned}$$

$$\begin{aligned}
&= (\rho_0 + g_0^2 + g_1^2)\left[(\rho_0 + g_0^2)(\rho_0 + g_3^2) - \rho_1^2\right] - \\
&\quad -(\rho_1 + g_0 g_1)^2(\rho_0 + g_3^2) + 2\rho_1\rho_2(\rho_1 + g_0 g_1) - \frac{\rho_1^2\rho_2^2}{(\rho_0 + g_3^2)} + \\
&\quad + \frac{\rho_1^2\rho_2^2}{(\rho_0 + g_3^2)} - \rho_2^2(\rho_0 + g_0^2) \\
&= (\rho_0 + g_0^2 + g_1^2)\left[(\rho_0 + g_0^2)(\rho_0 + g_3^2) - \rho_1^2\right] - \\
&\quad - \frac{1}{(\rho_0 + g_3^2)}\left[(\rho_1 + g_0 g_1)(\rho_0 + g_3^2) - \rho_1\rho_2\right]^2 - \\
&\quad - \frac{\rho_2^2}{(\rho_0 + g_3^2)}\left[(\rho_0 + g_0^2)(\rho_0 + g_3^2) - \rho_1^2\right] \\
&= \frac{\Delta_1}{\rho_0 + g_3^2} \quad .
\end{aligned}$$

From the expression of $\mathbf{A}_{dir}$, then, the matrix

$$[\mathbf{A}_{dir}^{-1}]_{1,2} = \frac{\rho_0 + g_3^2}{\Delta_1}\begin{bmatrix} (\rho_0 + g_0^2)(\rho_0 + g_3^2) - \rho_1^2 & -(\rho_1 + g_0 g_1)(\rho_0 + g_3^2) + \rho_1\rho_2 \\ -(\rho_1 + g_0 g_1)(\rho_0 + g_3^2) + \rho_1\rho_2 & (\rho_0 + g_0^2 + g_1^2)(\rho_0 + g_3^2) - \rho_2^2 \end{bmatrix} .$$

coincides with that in (B.7.27) ∎

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