FILTER DESIGN USING POLYNOMIAL EQUATIONS

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The polynomial approach to Wiener filter design will be discussed, using a simple prototype problem. An overview is also given over MSE (or H_2) filtering problems where this approach has been used. Furthermore, a new simple way of deriving the relevant polynomial equations will be presented.

1 Why use a polynomial approach to filtering?

Most of us, on some occation, have to design filters, predictors or fixed-lag smoothers, which minimize a mean square estimation error. Assume that relevant signals can be modelled as generated by linear time-invariant stochastic systems. Estimators can then be designed by the *Wiener approach* [1], as *Kalman filters* or by the *polynomial approach* [2]-[12]. The resulting estimators are equivalent, but the polynomial approach provides some advantages. In contrast to a Wiener frequency-domain design, it provides estimators parametrized by a finite number of parameters (polynomial coefficients). The method furthermore corresponds to a systematic numerical way of evaluating the causal factor of a Wiener-Hopf solution.

Compared to Kalman filters, the calculations for obtaining the estimators are often simpler, in particular for smoothing problems with coloured noise. (Calculation of the solution by hand is possible in low order problems.) Difficulties are also avoided in singular situations, i.e. when white noise is not present on all measurements. With the polynomial approach, estimators are obtained as transfer functions. Thus, classical filter concepts, such as frequency responses, poles and zeros can be studied directly.

2 A simple estimation problem

Consider estimation of a discrete-time signal in coloured noise. The signal $s(t)$ is scalar, but may be complex-valued. It is modelled as an ARMA-process

$$s(t) = -d_1 s(t-1) - \ldots - d_{nd} s(t-nd) + e(t) + c_1 e(t-1) + \ldots + c_{nc} e(t-nc) \quad .$$

Let q^{-1} be the backward shift operator ($q^{-1}s(t) = s(t-1)$). The signal can then be expressed in shift operator polynomial form as

$$s(t) = \frac{(1 + c_1 q^{-1} + \ldots + c_{nc} q^{-nc})}{(1 + d_1 q^{-1} + \ldots + d_{nd} q^{-nd})} e(t) = \frac{C(q^{-1})}{D(q^{-1})} e(t) \quad . \tag{2.1}$$

The signal $s(t)$ is to be estimated from noisy measurements

$$y(t) = s(t) + \frac{(1 + m_1 q^{-1} + \ldots + m_{nm} q^{-nm})}{(1 + n_1 q^{-1} + \ldots + n_{nn} q^{-nn})} v(t) = s(t) + \frac{M(q^{-1})}{N(q^{-1})} v(t) \tag{2.2}$$

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up to time $t+m$, using a stable linear estimator

$$\hat{s}(t|t+m) = \frac{Q(q^{-1})}{R(q^{-1})} y(t+m) \quad . \tag{2.3}$$

The problem formulation includes prediction ($m < 0$), filtering ($m = 0$) and fixed-lag smoothing ($m > 0$). The noises $e(t)$ and $v(t)$ are mutually independent and white. They have zero means and variances $\lambda_e > 0$ and $\lambda_v \geq 0$, respectively. The ARMA-models C/D and M/N are stable, causal and have no common zeros on the unit circle. The measurements $\{y(t)\}$ can also be described by the innovations model

$$y(t) = \frac{\beta(q^{-1})}{D(q^{-1})N(q^{-1})}\eta(t) \;=\; \frac{C(q^{-1})}{D(q^{-1})}e(t) + \frac{M(q^{-1})}{N(q^{-1})}v(t) \tag{2.4}$$

where the innovations sequence $\eta(t)$ has variance λ_η. For any polynomial $P(q^{-1}) = p_o + p_1 q^{-1} + \ldots + p_{np}q^{-np}$, define the *conjugate polynomial* $P_*(q) = p_o^* + p_1^* q + \ldots + p_{np}^* q^{np}$. Here, q is the forward shift operator and p_j^* is the conjugate of the (possibly complex) coefficient p_j. In the frequency domain, the complex variable z is substituted for q. *Stable* polynomials $P(z^{-1})$ have all zeros in $|z| < 1$. Polynomial arguments are often omitted below.

The monic polynomial $\beta(q^{-1})$ in (2.4) is the (polynomial) *spectral factor*. Use of (2.4) gives the spectrum of $y(t)$ as $\phi_y = \lambda_\eta \beta\beta_* / DND_*N_* = \lambda_e CC_*/DD_* + \lambda_v MM_*/NN_*$. Thus, $\beta(q^{-1})$ is seen to satisfy the *spectral factorization* equation

$$r\beta\beta_* = CC_*NN_* + \rho MM_*DD_* \tag{2.5}$$

where $r = \lambda_\eta/\lambda_e$ and $\rho = \lambda_v/\lambda_e$. Under the assumptions on the system, $\beta(z^{-1})$ is stable.

The mean square error (MSE) $E|\varepsilon(t)|^2$, where $\varepsilon(t) \triangleq s(t) - \hat{s}(t|t+m)$, is to be minimized. In Section 3, it will be shown that the optimal estimator is given by the IIR filter

$$\hat{s}(t|t+m) = \frac{Q_1 N}{\beta} y(t+m) \tag{2.6}$$

where $Q_1(q^{-1})$, together with a polynomial $L_*(q)$, can be found as the unique solution to the single *Diophantine equation*

$$q^{-m}CC_*N_* = r\beta_* Q_1 + qDL_* \quad . \tag{2.7}$$

Thus, the estimator is obtained by solving (2.5) for $\beta(q^{-1})$ (and r) and (2.7) for $Q_1(q^{-1})$ (and $L_*(q)$). The IIR-filter (2.6) is internally stable, since $\beta(q^{-1})$ is stable. It may contain stable common factors. Note that when the noise model has resonances (zeros of N close to the unit circle), the filter (2.6) has notches at the corresponding frequencies. Frequency weighting can be introduced into the design, by means of polynomial or rational filters in the criterion. See [12] or [16].

Closed-form expressions exist for second-order spectral factors [14]. For real-valued coefficients, the right-hand side of (2.5) is $g_o + g_1(q + q^{-1}) + g_2(q^2 + q^{-2})$. Then, r and $\beta(q^{-1}) = 1 + \beta_1 q^{-1} + \beta_2 q^{-2}$ will be given by

$$\gamma \triangleq \frac{g_o}{2} - g_2 + \sqrt{\left(\frac{g_o}{2} + g_2\right)^2 - g_1^2} \;;\; r = \left(\gamma + \sqrt{\gamma^2 - 4g_2^2}\right)/2 \;;\; \beta_1 = \frac{g_1}{r + g_2} \;;\; \beta_2 = \frac{g_2}{r}$$

There exist efficient numerical algorithms for polynomial spectral factorization. Some can be applied in multivariable problems as well, where $\beta(q^{-1})$ may be a matrix [15].

The linear equation (2.7) corresponds to a linear system of equations in the coefficients of $Q_1(q^{-1})$ and $L_*(q)$ See e.g. [7], [10], [11] or [12] for some examples. By setting the maximal degrees in q^{-1} and in q, respectively, equal on both sides of (2.7), the degrees of the unknown polynomials are obtained. The number of equations will equal the number of unknowns. The system of equations is nonsingular, since $\beta_*(z)$ (unstable) and $D(z^{-1})$ (stable) cannot have any common factors. Consequently, a unique solution exists.

Thus, the optimality requirement determines the *structure* and *degree* of the estimator. The methodology cannot be utilized for optimizing filters with a prespecified restricted complexity and degree. (A well-known closed-form expression exists for optimal FIR-filters. No corresponding expression is known for IIR-filters of fixed degree.)

Comparison with a Wiener solution reveals that solution of (2.5) corresponds to the calculation of a *whitening filter* (the inverse of the innovations model (2.4)). The calculation of the *causal part* $\{\cdot\}_+$ of the Wiener filter is performed by (2.7). See [12].

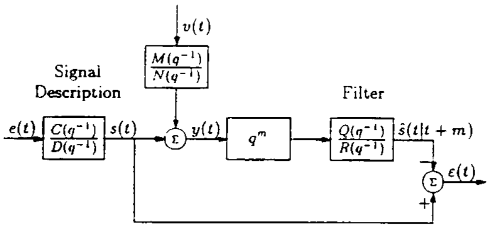
Multivariable generalizations of the problem above are treated in [4] and [6]. Estimation of internal states is described in [2], [3] and [6]. Input estimation, or deconvolution, has been treated in [7],[12] and [16], and adaptive algorithms have been developed [8]. A closed-form solution for the optimization of IIR filters in decision feedback equalizers (DFE's) was recently presented in [9]; the use of the polynomial approach was crucial for obtaining that result. Design of discrete-time differentiating filters, based on continuous-time or discrete-time signal models, has been investigated in [10] and [11].

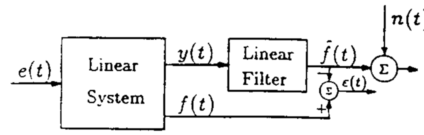
In most estimation problems of interest, the solution can be calculated from one polynomial (matrix) spectral factorization, followed by the solution of one linear polynomial (matrix) equation [2]. One exception is the optimization of DFE's described in [9]. It turns out to require no spectral factorization. Only a very simple linear system of equations needs to be solved. Another type of exception concerns signals described by strictly unstable models. Then, the solution of *two* coupled linear polynomial equations is sometimes required, to assure stationarity of the estimation error.

3 A novel technique for deriving design equations

Polynomial design equations for filtering and LQG control are traditionally derived using a "completing the squares"-argument. See e.g. [4],[5],[6] or [12]. In [12], a new and simpler constructive methodology is presented. It is based on the evaluation of orthogonality in the frequency domain, and leads to the goal in only a few well-defined steps of calculation. The problem considered is described by the right-hand figure below.

[2] In addition, *coprime factorizations*, which represent a kind of commutation operation for polynomial matrices, have to be performed in some multivariable estimation problems. See e.g. [12] or [16].





The signal $\hat{f}(t)$ is the estimate of the (possibly complex) signal $f(t)$, based on measurement of $y(t+m)$. The error $E|\varepsilon(t)|^2$ is to be minimized, where $\varepsilon(t) = f(t) - \hat{f}(t|t+m)$. The signal $n(t)$ is a variational term. The orthogonality principle requires the minimal estimation error to be orthogonal to any admissible variation $n(t)$, i.e. $E\varepsilon(t)n^*(t) = 0$. For scalar problems, this can be assured in the following way.

1. Parametrize the system by rational transfer functions. Define a polynomial spectral factorization from an innovations model of $y(t)$.

2. Introduce a variation of the estimate as $n(t) = \mathcal{G}(q^{-1})y(t+m)$. Here, $\mathcal{G}$ is an arbitrary transfer function, with the constraint that $\mathcal{G}$ should be causal and stable and $n(t)$ should be stationary. Express $E\varepsilon(t)n^*(t)$ in the frequency domain by means of Parseval's formula and simplify it, using the spectral factorization.

3. Fulfill the orthogonality requirement $E\varepsilon(t)n^*(t) = 0$ by *cancelling all poles inside the integration path by zeros.* This leads to the Diophantine design equation(s).

In the example of Section 2, **Step 1** is already completed. **Step 2** gives (see the left figure above) $E\varepsilon(t)n^*(t) = E(s(t) - \hat{s}(t))(\mathcal{G}y(t+m))^* =$

$$
\begin{aligned}
&= E\left(\frac{(R - q^m Q)C}{RD}e(t)\right)\left(\mathcal{G}q^m\frac{C}{D}e(t)\right)^* - E\left(q^m\frac{QM}{RN}v(t)\right)\left(\mathcal{G}q^m\frac{M}{N}v(t)\right)^* \\
&= \frac{\lambda_e}{2\pi j}\oint_{|z|=1}\frac{(z^{-m}RCC_*NN_* - Qr\beta\beta_*)}{RDD_*NN_*}\mathcal{G}_*\frac{dz}{z} \qquad (3.1)
\end{aligned}
$$

In **Step 3**, we note that the stable polynomials R, D and N have zeros in $|z| < 1$, while the poles of $\mathcal{G}_*$ and the zeros of D_*N_* are in $|z| > 1$. Thus, all poles inside $|z| = 1$ of the integrand of (3.1) are eliminated if (and only if)

$$z^{-m}RCC_*NN_* - Qr\beta\beta_* = zRDNL_*$$

for some polynomial $L_*(z)$. Now, N must be a factor of $Qr\beta\beta_*$. Set $Q = Q_1N$. Cancel N and substitute q for z:

$$R(q^{-m}CC_*N_* - qDL_*) = Q_1r\beta\beta_* \quad .$$

Evidently, R must be a factor of $Q_1r\beta\beta_*$. Since β_* is unstable, while Q_1 is part of the estimator numerator, set $R = \beta$. Thus, (2.6) and (2.7) are obtained.

This procedure is simple to apply also in multi-signal estimation problems [12] and in multivariable feedback and feedforward control problems, see [13]. Application to estimation problems with marginally stable signal models is discussed in [11].

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