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ORTHOGONALITY EVALUATED IN THE FREQUENCY DOMAIN: A NEW AND SIMPLE TOOL FOR DERIVING OPTIMAL IIR-FILTERS

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Abstract

A new method for deriving mean square optimal IIR - filters is presented. It is based on transfer function parametrizations and rests on orthogonality, evaluated in the frequency domain. In contrast to other, well known, methods it is simple to use. It is applicable to both scalar and multivariable filtering and control problems. The method is illustrated for the well known, scalar, output filtering problem. Beside this, a brief discussion on adaptive input estimation and equalization is included.

1 Introduction

In the project "Adaptive Wiener filters for control and signal processing", the main goal is to develop and apply methods useful to both fields. The main efforts are focused on equalization (or input estimation) and noise cancelling (or feedforward control). The work is intended to be of both theoretical and practical nature. In this paper, we will, in Section 2, present a new, simple and useful technique for deriving optimal IIR-filters. In Section 3, some problems which actualized the need of deriving optimal IIR-filters are discussed briefly. Possible future directions of research, leading to adaptive filters, are indicated.

2 Derivation of optimal IIR-filters

In many signal processing and control problems, it is desirable to obtain optimal filters or regulators. Normally quadratic criteria are minimized, since second order statistics is usually of interest. For filtering problems, using stationary input-output data, this is known as "Wiener filtering".

Using infinite future data, *nonrealizable Wiener filters* are straightforward to obtain. The solution is also straightforward in many problems when a *pre-specified FIR-filter* parametrization and a finite amount of future data is used. However, it is often unsatisfactory to be restricted to work either with optimal filters wich are not exactly realizable, or with realizable filters with a prespecified, and often suboptimal, FIR-structure. The problem of deriving *realizable, stable* and *explicit* solutions with more general IIR-filter parametrizations is far from trivial. (For example, it must be decided which filter structure to use for a particular problem.) For such problems, the following four methods are known:

1. The problem may be transformed to state space form. A (stationary) *Kalman filter* may then be designed.
2. The classical Wiener filtering approach is to use *variational arguments*, to obtain frequency functions whose causal parts, $\{\Sigma_{-\infty}^{\infty}\}_{+}$, are used. These parts are evaluated by residue calculus. See e.g. [2] and [3], chapter 13.
3. The polynomial approach, pioneered by Kučera [1], [10] provides filters directly in polynomial form. The equations defining optimality are derived by *"completing the squares"* in the quadratic criterion. Optimal filters are then designed by solving these equations. They usually consist of a spectral factorization and a single or two coupled linear polynomial equations.
4. Optimal filters in polynomial form can also be derived by *differentiating the criterion* with respect to the filter coefficients and ensuring that all sensitivity functions vanish [11].

In this context, the signal models are assumed to be given in transfer function form. While straightforward in principle, a detour via state space methods then tends to reduce the physical insight. State space solutions are also rather complicated, in particular when coloured noise is present and smoothing filters are sought. Use of the frequency domain/polynomial methods 2 - 4 above is unfortunately rather cumbersome. This applies especially for multivariable problems. See [1], [2] and [3].

Here, we will present a new derivation technique, to be used when optimal IIR-filters are sough. It leads to equations for calculating the filters which are equivalent to those derived from the polynomial approaches 3. and 4. above. The derivation technique is, however, much simpler, especially in the multivariable case. It basically rests on *orthogonality, evaluated in the frequency domain.* While the technique can be applied just as well in LQG control problems, we will, for clarity, restrict the discussion to estimation problems in this paper.

2.1 Outline of the technique

Assume a linear stochastic system to be parametrized by discrete time transfer functions and ARMA models. It generates a *measurement signal* $y(t)$ and a *desired response* $f(t)$, For simplicity, in the sequel, we assume these signals to be real-valued and scalar. Our aim is to optimize a linear filter which operates on $y(t+m)$ and estimates $f(t)$,

$$\hat{f}(t|t+m;\theta) \triangleq \frac{Q(q^{-1})}{R(q^{-1})} y(t+m) \tag{2.1}$$

Depending on m, (2.1) constitutes a predictor ($m < 0$), a filter ($m = 0$) or a fixed lag smoother $m > 0$. The estimator is designed to minimize the quadratic criterion

$$V(\theta) = E\varepsilon(t,\theta)^2 \tag{2.2}$$

under the constraint of causality and stability of the filter $Q(q^{-1})/R(q^{-1})$, where

$$\varepsilon(t,\theta) \triangleq f(t) - \hat{f}(t|t+m;\theta) \tag{2.3}$$

We require $\varepsilon(t,\theta)$ to be stationary. The polynomials $Q(q^{-1}) = Q_0 + Q_1 q^{-1} + ... + Q_{nQ} q^{-nQ}$ and $R(q^{-1}) = 1 + r_1 q^{-1} + ... + r_{nr} q^{-nr}$ have the backward shift operator ($q^{-1}y(t) = y(t-1)$) as arguments. The coefficients are collected in the parameter vector $\theta = (Q_o \dots Q_{nQ}\ r_1 \dots r_{nr})$. The minimizing argument of (2.2) is denoted θ^*. For any polynomial $P = P(q^{-1})$, we denote $P_* = P(q)$. In the frequency domain, the complex variable z is substituted for the forward shift opterator q, defining the stability region to be located inside $|z| < 1$.

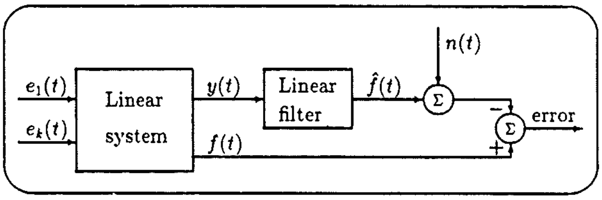

Figure 1. The filtering problem.

If we choose some filter Q/R, will it be possible to find a better one? Introduce an alternative estimator candidate , say $\hat{d}(t|t+m)$, such that

$$\hat{d}(t|t+m) = \frac{Q(q^{-1})}{R(q^{-1})} y(t+m) + n(t) \tag{2.4}$$

where $n(t)$ is an arbitrary stationary term, free to choose. It may depend on a linear combination [1] of measured data up to time $t+m$. For example, $n(t)$ may be described by $n(t) = (G/H)y(t+m)$. Substituting $\hat{d}(t|t+m)$ for $\hat{f}(t|t+m)$ in (2.3) gives the criterion

$$\overline{V}(\theta) = E\varepsilon(t)^2 - 2E\varepsilon(t)n(t) + En(t)^2 \tag{2.5}$$

Depending on the choice of $n(t)$, $\overline{V}(\theta)$ might attain a greater or lower value, compared to $V(\theta)$. The *key idea* is now to choose the filter polynomials Q and R such that $\varepsilon(t)$ becomes orthogonal to *any* admissible additional signal $n(t)$. This means that *the mixed term in* (2.5) *is set to zero.* This condition will be sufficient to determine the polynomial degrees and the coefficient values of Q and R in (2.1), (2.4) *uniquely.* With Q and R chosen such that $E\varepsilon(t)n(t) = 0$, it is then obvious that the choice $n(t) \equiv 0$ will minimize (2.5) and therefore also (2.2). Since no modification of the filter can improve the criterion value, the derived Q and R polynomials are optimal and $E\varepsilon(t,\theta^*)^2$ is the minimal value.

Since $\varepsilon(t)$ and $n(t)$ are stationary, the mixed term in (2.5) can be expressed with the aid of Parsevals formula

$$E\varepsilon(t)n(t) = \frac{1}{2\pi j} \oint_{|z|=1} \phi_{\varepsilon n} \frac{dz}{z} \tag{2.6}$$

where $\phi_{\varepsilon n}$ is a rational function in z and z^{-1} which depends on the filter polynomials Q and R. Let $\phi_{\varepsilon n}$ be defined as

$$\phi_{\varepsilon n} = \frac{T(z,z^{-1})}{S_+(z,z^{-1})S_-(z,z^{-1})} \tag{2.7}$$

where the polynomials S_+ and S_- have all zeros strictly inside and outside the unit circle, respectively. In order to force (2.6) to zero, we require

$$T(z,z^{-1}) = zL_*(z)S_+(z,z^{-1}) \tag{2.8}$$

where $L_*(z)$ is an arbitrary polynomial *only* in z (not in z^{-1}). Then, the integrand $L_*(z)/S_-(z,z^{-1})$ in (2.6) has no poles inside the integration path, and the integral vanishes.

When the measurable signal (or signals) is a sum of independent stochastic sequences, for example,

$$y(t) = \frac{C(q^{-1})}{D(q^{-1})} e(t) + \frac{M(q^{-1})}{N(q^{-1})} v(t) \tag{2.9}$$

a spectral factorization[2] must often be introduced:

$$r\beta\beta_* = CC_*NN_* + \rho DD_*MM_* \quad ; \quad \rho = \lambda_v/\lambda_e \tag{2.10}$$

In (2.10), r is a scalar and β is the stable spectral factor. It is monic by construction. Expressing (2.7) in terms of the spectral factorization and (2.8), leads to one or two coupled linear polynomial equations in $Q(z^{-1})$, $R(z^{-1})$ and $L(z)$. They will be polynomial equations in both z and z^{-1}. If a solution exists, it will be unique. For open loop problems, such as filtering, equalization and noise cancelling problems, we will end up with a single polynomial equation. The filter denominator R will, in such cases, be equal to the spectral factor β, or have β as a factor.

The derivation procedure can be summarized as follows.

1. Define (if needed) one or several spectral factorizations.
2. Calculate $\varepsilon(t)$ in terms of the involved polynomials and add a signal $n(t)$ to the filter output. Use Parsevals formula to express the mixed term in (2.5), using the spectral factor(s).
3. Cancel the denominator poles of (2.7) inside the unit circle by means of factors in the numerator. This leads to one or two coupled polynomial equations, to be solved for the estimator polynomials.

[1] Since we consider only linear filters, the additional term $n(t)$ should be a *linear* combination of data. Since the filtering error must be *stationary*, nonstationary $n(t)$ are excluded.

[2] In some problems, such as in the derivation of the optimal DFE presented in [4], [5], the spectral factor is not needed. This situation appears when measurements are available in every branch driven by a single noise source.

We now illustrate the procedure for a simple example.

2.2 Example: The scalar output filtering problem.

Let the output measurements be described by (2.9), where D and N are stable. The signal $s(t) = (C/D)e(t)$, depicted in Figure 2, is to be estimated.

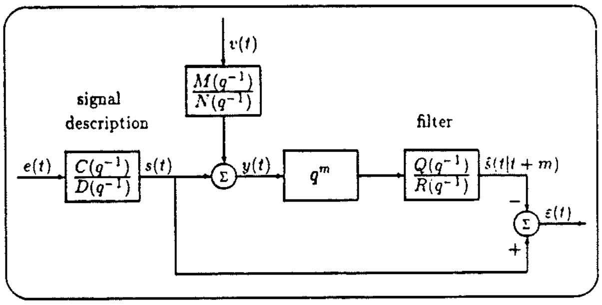

Figure 2. The scalar output filtering problem ($m = 0$). The signal $s(t)$ is to be estimated from $y(t)$. ($\rho = \lambda_v/\lambda_e$).

The spectral factorization (2.10) is easily derived from the innovations model of $y(t)$. It is assumed to be stable, i.e. the right-hand side has no zeros on the unit circle. We seek a stable causal linear filter Q/R which minimizes $E\varepsilon(t)^2$.

The error signal is given by

$$\varepsilon(t) = \frac{C(R-Q)}{DR}e(t) - \frac{MQ}{NR}v(t) \tag{2.11}$$

and the additional, arbitrary signal is

$$n(t) = \frac{G}{H}y(t) = \frac{G}{H}\left(\frac{C}{D}e(t) + \frac{M}{N}v(t)\right) \tag{2.12}$$

with G and H undetermined, but H stable. The mixed term in (2.5) is then directly found to be

$$E\varepsilon(t)n(t) =$$

$$= E\left(\frac{C(R-Q)}{DR}e(t)\frac{CG}{DH}e(t)\right) - E\left(\frac{QM}{RN}v(t)\frac{MG}{NH}v(t)\right)$$

$$= \frac{\lambda_e}{2\pi i}\oint_{|z|=1}\left(\frac{(R-Q)CC_*NN_* - \rho QMM_*DD_*}{RDD_*NN_*}\right)\frac{G_*}{H_*}\frac{dz}{z}$$

$$= \frac{\lambda_e}{2\pi i}\oint_{|z|=1}\frac{(RCC_*NN_* - Qr\beta\beta_*)}{RDD_*NN_*}\frac{G_*}{H_*}\frac{dz}{z}$$

In the second equality, Parsevals formula was used and in the last, the spectral factorization (2.10) was inserted. Thus, we have completed steps 1 and 2 and now proceed with step 3. Since R, D, N and H are assumed to be stable polynomials, they will have all their zeros inside the unit circle (D_*, N_* and H_* will have all zeros outside the unit circle!) The numerator of the integrand (2.7) then fulfills (2.8) if and only if

$$RCC_*NN_* - Qr\beta\beta_* = zL_*RDN \tag{2.13}$$

Obviously, Q must have N as a factor, $Q = Q_1N$. Rearranging the factors in (2.13) gives

$$R(CC_*N_* - zL_*D) = Q_1r\beta\beta_* \tag{2.14}$$

Since R must be stable and $z^{-n\beta}\beta_*(z)$ is unstable while Q_1 is a filter polynomial, choose $R = \beta$. Then, Q_1, together with L_*, can be found as the solution to the linear polynomial equation

$$CC_*N_* = r\beta_*Q_1 + zDL_* \tag{2.15}$$

with degrees

$$nQ_1 = \max(nc, nd+1) \qquad nL_* = n\beta - 1 \tag{2.16}$$

Equation (2.15) is solvable since $D(z^{-1})$ (stable) and $z^{-n\beta}\beta_*(z)$ (unstable) cannot have common factors. The solution (Q_1, L_*) is unique.

Since the second term in (2.5) becomes zero and the remaining, third term, is quadratic, the optimal choice of $n(t)$ is zero. Obviously the solutions above is then optimal. The filter is calculated by solving (2.10) for β, and then solving (2.15) for Q_1 and L_*.

2.3 Remarks and interpretations

Note that causality [3] of the filter is guaranteed by requiring Q and R to be polynomials in the *backward* shift operator q^{-1}. The *stability* of the filter is simple to assure, with the unique choice $R = \beta$ in (2.14). This choice is unique in the sense that any other choice ($\beta = \beta_1\beta_2, R = \beta_1, \deg\beta_2 > 0$) would make the polynomial equation resulting from (2.14) unsolvable, unless β_2 were a common factor of C and D.

The polynomial L_* does not appear in the filter. It is, however, important that L_* should be a polynomial in q (or in z), *not* in q^{-1} (or z^{-1}). If L_* contained the argument z^{-1}, the integrand (2.7) would have poles in the origin, and the integral would not vanish. The requirements that L_* should be a polynomial only in z, while Q_1 is a polynomial only in z^{-1}, lead to a unique solution to the linear polynomial equation (2.15), with degrees (2.16). (These degrees are determined by the requirement that $L_*(z)$ should cover the maximal occuring power of z, while $Q_1(z^{-1})$ covers the maximal power of z^{-1}.) This gives a linear system of equations, with equal number of equations and unknowns.

The derivation technique provides a solution with an *optimal filter structure*. Above, it is an IIR-filter Q_1N/β. It is important to postulate a filter with the most general structure from the outset. If we had specified the filter to be FIR ($R = 1$), equation (2.14) would have been unsolvable.

A filter attains the minimal estimation error if and only if it contains the same coprime factors as the filter derived above. The solution is thus *unique*, modulo possible stable common factors in the filter. (If, for example, D and N have common factors, these factors will appear in β. They are thus common factors of the filter Q_1N/β.)

[3] "Causality" does not exclude consideration of smoothing problems. It only implies that optimal filters do not require data further into the future than defined by the prescribed smoothing lag m.

The technique can also be applied to systems where $y(t)$ and/or $f(t)$ are *nonstationary*, under some technical conditions which correspond to detectability. The important point is that the signals $\varepsilon(t)$ and $n(t)$, which appear in the criteria (2.2) and (2.5), must be stationary.

The idea of adding an arbitrary term, $n(t)$, to the filter Q/R in (2.4), is useful also when candidates for optimal filters (Q/R) have been derived by other means. The idea is then to prove, by contradiction, that $n(t)$ must be zero. This approach, originally used in [12], has been applied in some of our earlier work [4],[5],[7],[9],[11]. In contrast to this, the derivation technique presented above is a *constructive* way to find optimal IIR-filters.

The suggested derivation procedure also applies to *multivariable* filtering and control problems. As an example, the multivariable filtering result given in [6] is derived in just a few lines with the new method. The need for a simple derivation technique, as e.g. described above, has been actualized when investigating explicit and optimal solutions to some filtering problems described below.

3 Adaptive input estimation and equalization

In [4] and [5], an optimal and explicit IIR - decision feedback equalizer was derived for a general channel structure with coloured noise. For practical reasons, it is desirable to create an adaptive scheme. In order to do this, an adaptive input estimator has been derived and tested in [7]. Under certain identifiability conditions, given in [8], the linear deconvolution problem described in [9], has been solved on-line. The results in [7], based on an explicit scheme, are promising for adaptive DFE's and LFE's under certain circumstances.

Adaptive equalization based on the DFE described in [4], [5] may be developed along the following lines.

1. This step is a common part to 2a, 2b. The channel structure introduced in [4], [5] is depicted in Figure 2. In "reference mode", $e(t)$ is a known sequence. During this period, (C, D, M, N) are estimated by means of system identification. Based on this channel estimate, the DFE described in [4], [5] is calculated. This DFE is used as an initial estimate to step 2, the "equalization mode".

2a. The results in [7] is adopted. Under certain conditions on the channel and measurement noise [8], the (C, D, M, N) polynomials can be estimated adaptively, from the received sequence only. Based on these estimates, the DFE is updated.

2b. Direct adaptation of the filters in the IIR-DFE structure in [4], [5]. In contrast to step 2a, the equalizer update is then based on decisioned data.

Which line to follow is a topic for further research. It can be noted, though, that the basic principle to be used in 2a. works well, as reported in [7]. It is thus reasonable to assume that it will work for the equalization problem if, loosely speaking, the channel and measurement noise is not too extreme.

4 Conclusions

A new method for deriving mean square optimal IIR-filters has been presented. It is transfer-function based and utilizes orthogonality evaluated in the frequency domain, constructively. The method is simple and straightforward to use and applies to scalar and multivariable prediction, filtering or smoothing problems. The optimal filters are obtained by solving a spectral factorization and a polynomial equation. A simple example illustrates the main ideas. The need for deriving optimal IIR-filters has been actualized when investigating e g optimal equalizers. We have briefly discussed future directions of research on adaptive equalizers.

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