ON THE OPTIMALITY AND PERFORMANCE OF TRANSMIT AND RECEIVE SPACE DIVERSITY IN MIMO CHANNELS

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Abstract

The issue of channel state information at the transmitter is investigated using MIMO channel measurements and by deriving expressions for ergodic and outage capacity in a Rayleigh fading channel. Expressions for bit error rates in Rayleigh fading channels are also presented for orthogonal space time block codes and for beamforming where the bit error rates in the beamforming case follow from the distribution of the largest eigenvalue to Wishart matrices. We demonstrate by measurements that Rayleigh fading is a valid assumption in non line of sight channels, although a Nakagami-m distribution showed to be a more appropriate distribution model in both line of sight and NLOS environments. It was also demonstrated that channel state information at the transmitter is less useful in high-SNR scenarios but is more useful in line of sight channels compared to non line of sight environments.

1 Introduction

This paper deals with multiple input multiple output (MIMO) wireless systems in flat fading channels. The aim is to answer the question: How large is the performance gain by providing the transmitter with channel state information (CSI) ?. CSI can be made available to the transmitter by using a feedback channel[1] which will consume bandwidth and is thereby undesirable if the net gain is small. When some or full CSI is available at the transmitter, beamforming maximizes the receiver SNR [1]. When CSI is unavailable at the transmitter, space time codes [2] can be used to achieve transmit diversity gain over the wireless link. In this paper we compare the beamforming approach to the space time block codes using both a theoretical channel model and measured channels. In the theoretical model, we assume a flat Rayleigh fading channel and we use the measurements to validate this assumption.

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[1]The delay between acquiring the CSI to this information is used must of course be shorter than the coherence time of the channel.

2 Data and channel model

Assume a wireless link using a transmitter with $n_t > 1$ antennas and a receiver with n_r antennas. The $n_r \times n_t$ flat fading channel gain is described by the matrix $\mathbf{H}$. The i, j^{th} element of $\mathbf{H}$ is thus the complex gain factor between receive antenna i and transmit antenna j. Furthermore, the communication is carried out using bursts or packets of length T symbols, and we assume that the channel is quasi-static, i.e. the elements of $\mathbf{H}$ are assumed to be fixed during the transmission of these T symbols. The input output relation can be written as

$$\mathbf{Y} = \mathbf{HC} + \mathbf{V} \tag{1}$$

where the received signal $\mathbf{Y}$ is $n_r \times T$, the encoded codeword $\mathbf{C}$ is $n_t \times T$ and the receiver noise $\mathbf{V}$ is $n_r \times T$ and the elements of $\mathbf{V}$ are independent and identically distributed (i.i.d.) zero mean circular complex Gaussian random variables with variance σ^2. Furthermore, we have a constraint on the maximum transmitted power on n_t antennas, P_T. Define $P = \frac{P_T}{n_t\sigma^2}$.

The mutual information between the receiver and the transmitter for the channel model (1) is [3]

$$I(\mathbf{H}, \mathbf{R}_{cc}) = \log_2 \det \left[\mathbf{I}_n + \frac{1}{\sigma^2}\mathbf{H}\mathbf{R}_{cc}\mathbf{H}^*\right] \tag{2}$$

measured in bits per second per Hz of bandwidth. $\mathbf{R}_{cc}$ is the covariance matrix of the transmit signal with the power constraint Trace$(\mathbf{R}_{cc}) = P_T$ and * denotes complex conjugate transpose. If CSI is unavailable at the transmitter, capacity is achieved by choosing the transmit data streams as circularly symmetric zero mean complex Gaussian variables with covariance matrix $\mathbf{R}_{cc} = P_T/n_t\mathbf{I}$ [3]. If the channel is random, we define the ergodic capacity as $C = E\{I\}$.

2.1 Space time block codes

Space time block codes (STBC) is a technique to map K input data symbols to the elements of the matrix $\mathbf{C}$. In [4], it was shown that the STBC decouples the MIMO channel into an equivalent SISO channel, as seen by each transmitted symbol, so the output can be written as (before ML detection)

$$z_o = \|\mathbf{H}\|_F^2 \sqrt{P_T/n_t}s + v \tag{3}$$

where s is the transmitted symbol normalized to unit magnitude and v is the noise whose variance can be shown to be $\|\mathbf{H}\|_F^2 \sigma^2$. The effective SNR at the receiver can thus be shown to be[4]

$$\gamma_{STBC} = P \|\mathbf{H}\|_F^2 = P\gamma_H. \quad (4)$$

The outage capacity R_0 at probability P_{out} is given by solving for R_0 in

$$P_{out} = \Pr\{I < R_0\} = \quad (5)$$

$$= \Pr\left\{\frac{K}{T}\log_2(1 + P\gamma_H) < R_0\right\} = \quad (6)$$

$$= \int_0^{P^{-1}\left(2^{R_0T/K}-1\right)} p_H(\gamma)d\gamma \quad (7)$$

where $p_H(\gamma)$ is the probability density function of γ_H. Since the squared Frobenius norm of $\mathbf{H}$ is a sum of $2n_rn_t$ independent χ^2-distributed variables (under the iid Rayleigh fading assumption) , $p_H(\gamma)$ is also a χ^2-distribution [5]

$$p_H(\gamma) = \frac{1}{\Gamma(n_rn_t)}\gamma^{n_rn_t-1}e^{-\gamma} \quad (8)$$

in the Rayleigh fading case and non-central χ^2- distributed in the Ricean fading case. The expectation value of γ_H is $E(\gamma_H) = n_rn_t$. Using (8) in (7) we get

$$P_{out} = \Gamma\left(n_rn_t, P^{-1}\left(2^{R_0T/K} - 1\right)\right) \quad (9)$$

where $\Gamma(a, x) = \int_0^x t^{a-1}e^{-t}dt/(t-1)!$ is the incomplete Gamma function.

The ergodic channel capacity is the expectation value of the mutual information (2) and in the STBC case, since the log-function is concave, an upper bound can be found by using Jensen´s inequality

$$C_{STBC} = E\{I\} = E\left\{\frac{K}{T}\log_2(1 + P\gamma_H)\right\} \quad (10)$$

$$\leq \frac{K}{T}\log_2(1 + PE\{\gamma_H\}) = \frac{K}{T}\log_2\left(1 + \frac{P_T}{\sigma^2}n_r\right) \quad (11)$$

bps/Hz. Hence, the ergodic capacity upper bound increases logarithmically with the number of receive antennas in the STBC case.

The bit error rate for STBC can be calculated for coherently detected BPSK by performing the following expectation integral

$$P_{BPSK,STBC} = \quad (12)$$

$$= \int_0^\infty Q(\sqrt{2P\gamma})p_H(\gamma)d\gamma = \quad (13)$$

$$= \frac{1}{2} - \sqrt{\frac{P}{\pi}}\frac{\Gamma(n_rn_t + 1/2)}{\Gamma(n_rn_t)}F\left(\frac{1}{2}, \frac{1}{2} + n_rn_t; \frac{3}{2}; -P\right) \quad (14)$$

where $F()$ is the hypergeometric function [6] and $\Gamma()$ is the Gamma function. A closed form expression exists for this particular hypergeometric function [6].

2.2 Beamforming

If CSI is available at the transmitter, we use beamforming as it maximizes the receive SNR [1]. The input, output relation is

$$z_c = \mathbf{w}_R^*\mathbf{H}\mathbf{w}_T s + \mathbf{w}_R^*\mathbf{v} \quad (15)$$

where $\mathbf{w}_R$,$\mathbf{w}_T$ are the receive and transmit weight vectors respectively. The received SNR can be optimized by choosing $\mathbf{w}_R$,$\mathbf{w}_T$ as the principal left and right singular vectors to the matrix $\mathbf{H}$ respectively under the constraint $|\mathbf{w}_T|^2 = P_T$. The corresponding receive SNR is

$$\gamma_{BF} = \lambda_{max}\frac{P_T}{\sigma^2} = \lambda_{max}Pn_t \quad (16)$$

where λ_{max} is the largest eigenvalue to the complex *Wishart matrix* $\mathbf{W} = \mathbf{H}\mathbf{H}^*$. To proceed as in the previous section for the STBC case, we need the probability density function of λ_{max}.

2.2.1 Eigenvalues of W

Results from random matrix theory gives us the probability density function (pdf) of the eigenvalues of $\mathbf{W}$. The pdf of the largest eigenvalue to the matrix $\mathbf{W}$ can now be derived. Start with the $m = \min\{n_r, n_t\}$ unordered eigenvalues to the matrix $\mathbf{W}$ [7]

$$p(\lambda_1, \ldots, \lambda_m) =$$

$$= K_{m,n}\prod_{i=1}^m \lambda_i^{n-m}e^{-\lambda_i}\prod_{1\leq i<j\leq m}(\lambda_i - \lambda_j)^2 \quad (17)$$

where the constant $K_{n,m}$ is a normalization constant that depends on $n = \max(n_r, n_t)$ and m.

The pdf for the largest eigenvalue is obtained by first finding the cumulative distribution function as

$$Pr\{\lambda_{max} < t\} =$$

$$= \int_0^t \cdots \int_0^t p(\lambda_1, \ldots, \lambda_m)d\lambda_1 \cdots d\lambda_m \quad (18)$$

and then the pdf for λ_{max} as

$$p_\lambda(\lambda_{max}) = \frac{d}{dt}Pr\{\lambda_{max} < t\}\Big|_{t=\lambda_{max}}. \quad (19)$$

By inserting (17) and (18) into (19), it is straightforward to show that this pdf can be written as

$$p_\lambda(\lambda_{max}) = \sum_{k=1}^m \phi_k(\lambda_{max})e^{-k\lambda_{max}} \quad (20)$$

where $\phi_k(x)$ is a polynomial. The polynomials for $m, n = 2, 3$ are given in Appendix A.

The pdf of λ_{max} for the general n_r, n_t case is difficult to calculate but in the limit of $n_r, n_t \to \infty$ where $n_r/n_t \to \kappa \geq 1$ the expectation value of λ_{max} converges to $\left(\sqrt{n_r} + \sqrt{n_t}\right)^2$ [8].

The outage capacity can be found by carrying out a similar integration as in the STBC case (7), and the result will depend on the dimensions of the system through n_r, n_t. By using Jensen´s inequality and the asymptotic value for the expectation of λ_{max} we get an upper bound on the ergodic channel capacity

$$C_{BF} = E\{I\} = \tag{21}$$

$$\leq \log_2\left(1 + \frac{P_T}{\sigma^2} E\{\lambda_{max}\}\right) \tag{22}$$

$$\overrightarrow{asym} \log_2\left(1 + \frac{P_T}{\sigma^2}\left(\sqrt{n_r} + \sqrt{n_t}\right)^2\right) \tag{23}$$

bps/Hz.

We now derive the BER for the beamforming system assuming coherent BPSK. The average probability of bit error is then given by evaluating $E\left\{Q(\sqrt{2Pn_t\lambda_{max}})\right\}$.

$$P_{BF} = \int_0^\infty Q\left(\sqrt{2Pn_t\lambda_{max}}\right) p_\lambda(\lambda_{max}) d\lambda_{max} \tag{24}$$

and by integration we get the expression

$$P_{BF} = \frac{1}{2}\left(1 - \sum_{k=1}^{m} \sqrt{\frac{P_T/\sigma^2}{P_T/\sigma^2 + k}} \varphi_k^{(m,n)}(P_T/\sigma^2)\right) \tag{25}$$

where $\varphi_k^{(m,n)}(x)$ is a rational polynomial of degree m, shown in Appendix B for $n, m = 1, 2, 3$. Note that in the $n_t = n_r = 1$ single input single output case we get the classical expression for BER of coherent BPSK in a Rayleigh fading channel

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{P_T/\sigma^2}{P_T/\sigma^2 + 1}}\right) \quad . \tag{26}$$

It is straightforward to show that in the high SNR limit $P_T/\sigma^2 \gg 1$ we can approximate the BER in (25) with the expression

$$P_{BF} \approx \frac{K_C}{(P_T/\sigma^2)^{nm}} \tag{27}$$

where K_C is a constant. Hence, a diversity order of nm is achieved.

3 Optimal transmission

It is interesting to compare the capacities for the STBC and beamforming cases to what is optimally achievable in the two cases of CSI available and not available at the transmitter. In this section we briefly review the mutual information in the optimal case and also find an expression for when beamforming capacity equals waterfilling capacity.

3.1 CSI at the transmitter

The mutual information as a function of the ordered eigenvalues $\lambda_1 > \lambda_2 > \cdots \lambda_m$ can be written as

$$I_{wf} = \sum_{i=1}^{m} \log_2\left(1 + \frac{p_i}{\sigma^2}\lambda_i\right) \tag{28}$$

bps/Hz, where p_i is the power transmitted in the channel “mode” i with power gain λ_i. This is the upper bound by Shannon that gives the realizable information rate through parallel channels with additive white Gaussian noise. The maximum mutual information under the power constraint $P_T = \sum_{i=1}^{m} p_i$ is achieved by “water-filling” over the m channel modes [9] and the allocated power for mode i is

$$p_i = \left(\nu - \frac{1}{\lambda_i}\right)^+ \tag{29}$$

where $(x)^+ = \max(0, x)$.

When the ”water level” μ is smaller than the inverse of the second largest eigenvalue, $(\mu < \lambda_2^{-1})$, beamforming (which utilizes only the principal mode corresponding to λ_{max}) becomes equal to the waterfilling solution. Since μ also depends on $\lambda_1 = \lambda_{max}$, this occurs if

$$\frac{1}{\lambda_2} - \frac{1}{\lambda_1} < \frac{P_T}{\sigma^2} \tag{30}$$

which gives

$$I_{wf} = I_{BF} = \log_2\left(1 + \frac{P_T}{\sigma^2}\lambda_1\right) \quad . \tag{31}$$

Hence, only the first mode is “filled with water” or mathematically, $p_i = 0$ for $i = 2, \ldots, m$.

The probability of this to occur is

$$Pr\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} < \frac{P_T}{\sigma^2}\right) = $$

$$= \int_0^\infty \int_0^{\frac{\lambda_1}{P_t\lambda_1/\sigma^2 + 1}} p_o(\lambda_1, \lambda_2) d\lambda_2 d\lambda_1 \tag{32}$$

where $p_o(\lambda_1, \lambda_2)$ is the marginal pdf of the two largest eigenvalues which we find by integrating out the $m-2$ smaller eigenvalues in (17). The result can be seen in Figure 1 for different dimensions of the MIMO system in a Rayleigh fading channel. Note that only the smallest 2×2 system has a non-negligible probability to achieve the waterfilling capacity for practical SNR:s.

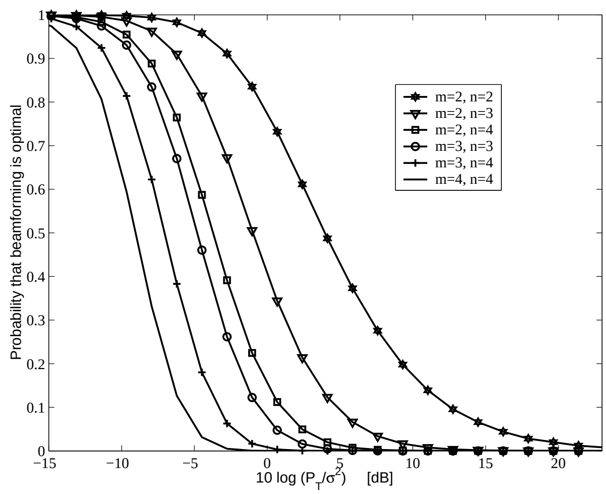

Figure 1: The probability that beamforming achieves waterfilling capacity in a Rayleigh fading channel.

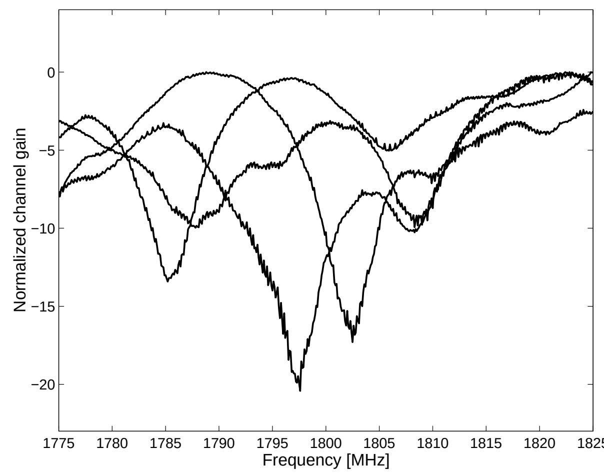

Figure 2: Power spectrum for NLOS channel. Received power in four half-a-wavelength spaced antennas from one transmit antenna.

3.2 No CSI at transmitter

Without CSI at the transmitter the mutual information is given by (28) with all $p_i = P_T/n_t$, hence an equal amount of power is transmitted in each channel "mode". The outage and ergodic capacity can now be calculated by using the pdf of the eigenvalues (17). An expression for the ergodic capacity was derived in [10] where also an upper and lower bound was presented.

4 Comparisons with measurements

In this section we compare the above the theoretical results presented above with MIMO channel measurements. An important issue is the validity of the Rayleigh fading channel assumption.

4.1 The Measurement Setup

The measurements were performed in an indoor office environment using a 4×4 MIMO system at the frequency 1.8 GHz. A Vector Network Analyzer was used to measure the channel coefficients for the 16 channels using a switching method. All 16 channels were measured in less than 3 seconds. Between each measurement, the array was moved one eight of a wavelength in the broadside direction. The antenna elements were microstrip patch antennas placed in a linear array with an interelement spacing of half a wavelength. The patch antennas had a half power beamwidth of 80° and a half power bandwidth of 170 MHz . Two scenarios were investigated, one line of sight (LOS) and one non-LOS (NLOS) setup. In the LOS scenario, the two arrays were placed facing each other in a 8×6 meter laboratory room containing various instruments, tables and cabinets and 146 measurements were conducted. In the NLOS scenario, 256 measurements was conducted and the receive array was placed outside the laboratory room, centered in a long corridor with the array broadside parallel with the corridor. The transmit array was kept in the adjacent laboratory.

4.2 Validation

The initial measurements aimed to verify the flat Rayleigh fading assumption. Figure 2 shows the measured power spectrum in the NLOS case from one transmit antenna to the four receive antennas. The coherence bandwidth is estimated to $B_c = 2.8$ MHz so the flat fading assumption is valid if the signalling bandwidth is less than B_c. If a system with higher bitrate is required, transmission over many subchannels can be used, where the bandwidth of each subchannel is less than B_c. Figure 3 shows the estimated probability density functions of the normalized amplitudes in the LOS and NLOS cases. The curves are fitted to a Nakagami -m distribution using a moment based method [11]. The Rayleigh distribution is a special case of the Nakagami-m distribution when $m = 1$. Using the method in [11], we estimate $m = 0.85$ in the NLOS case and $m = 5.54$ in the LOS case. Hence, the fading amplitude in the NLOS case is approximately Rayleigh distributed and the phase is uniformly distributed (not shown).

4.3 The benefits of feedback

All the plots in this section are derived from measurements if not stated otherwise. To analyze the performance gains from using feedback, the ergodic channel capacity in the optimal cases are compared in Figure 4. The benefits of CSI is decreased for increasing transmit power (or reduced receiver noise).

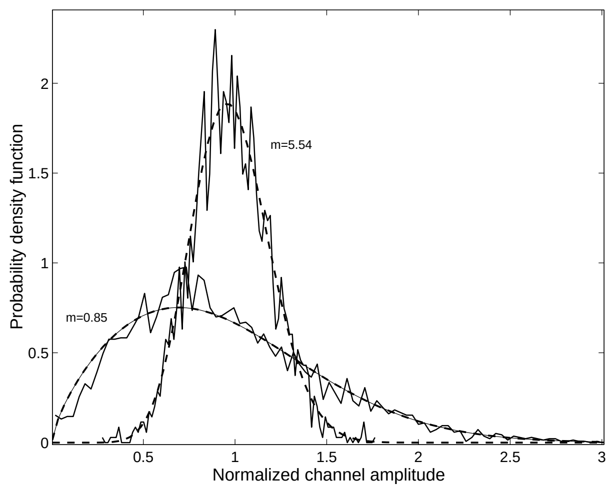

Figure 3: Estimated probability density functions of channel amplitudes, from measured data in LOS and NLOS channels. The dotted line shows the moment-based estimated Nakagami-m distribution. The m-parameters were estimated to 0.85 and 5.54 respectively.

We also see that CSI gives a larger improvement in the LOS case, because now there exists a strong mode in the channel which is more efficiently exploited by the waterfilling algorithm. Figure 5 shows

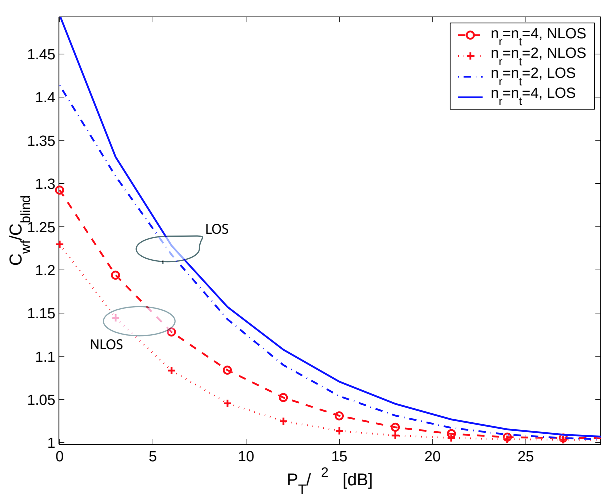

Figure 4: Relative ergodic capacity gain by using CSI as a function of transmitted power. Waterfilling versus optimal blind transmission.

the corresponding comparison but for the beamforming versus STBC case and the outage capacity is plotted. For two antenna element arrays, the difference between LOS and NLOS channels is small. Figure 6 shows the suboptimality of using beamforming when CSI is known at the receiver. Only for $n_t = n_r = 2$ MIMO systems at low SNR, the beamforming approach becomes equivalent to the waterfilling approach. The probability for this to occur is given by expression (32). It is instructive to

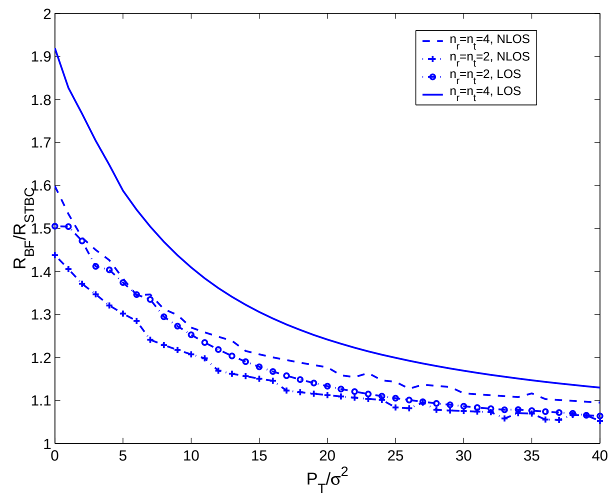

Figure 5: Relative outage capacity gain by using CSI as a function of transmitted power at $P_{out} = 0.1$ probability. . Beamforming versus space time block coding with $K{=}T$.

compare the BER also, as it is a good measure on how real systems behave with and without CSI. Figure 7 shows the BER estimated using the measured channels along with the theoretical curves from (25) and (14) for beamforming and STBC respectively. Due to the short measured data series, the measured curves cannot accurately estimate BER below 10^{-2}. This is the reasons for the deviation from the theoretical curves in Figure 7. For low SNR however, the number of bit errors are so large that the theoretical and measured curves coincide.

5 Conclusion

Performance of a MIMO systems using CSI and no CSI at the transmitter has been compared. Measurements at 1.8 GHz showed that the NLOS indoor channel amplitude is closely approximated by a Rayleigh fading distribution and in the LOS we can use a Nakagami -m distribution. It was demonstrated that the usefulness of CSI decreases when SNR is increased but increases when the channel becomes LOS.

When CSI is available at the transmitter, beamforming is always optimal if the number of receive antennas is one (since we only have one channel eigenvalue). In a MIMO system, beamforming is suboptimal except in the low SNR 2×2 case. A LOS channel reduces the difference between waterfilling and beamforming, since we then have a strong channel mode $\lambda_1 \gg \lambda_2 > \lambda_3 > \cdots$.

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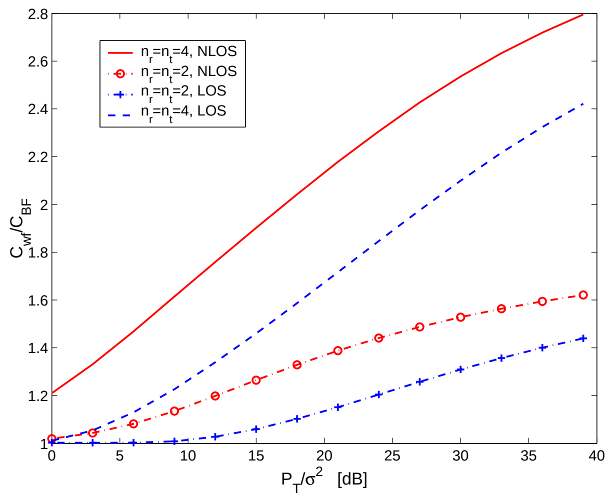

Figure 6: Relative ergodic capacity gain by waterfilling versus beamforming as a function of transmitted power .

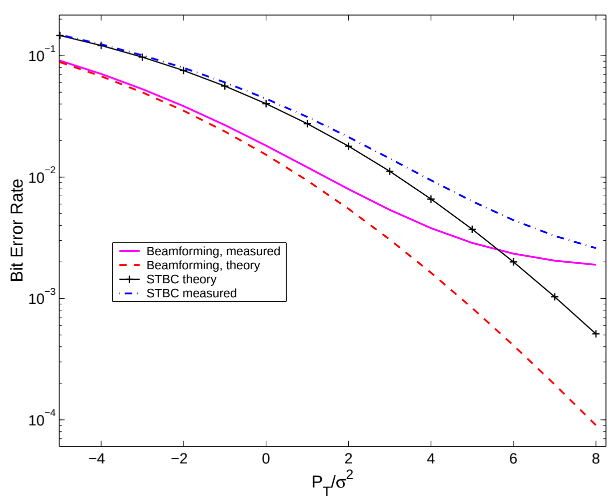

Figure 7: Bit error rates for beamforming and STBC ($n_r = n_t = 2$) as a function of transmitted power.

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A The polynomials $\phi_k(x)$

m=2,n=2 : $\phi_1(x) = 2 - 2x + x^2$ and $\phi_2(x) = -2$

m=2,n=3 : $\phi_1(x) = 3x - 2x^2 + x^3/2$ and $\phi_2(x) = -3x - x^2$

m=3,n=3 : $\phi_1(x) = 3 - 6x + 6x^2 - 2x^3 + x^4/4$, $\phi_2(x) = -6+6x-3x^2-x^3-x^4/2$ and $\phi_3(x) = 3$

B The rational polynomials $\varphi_k^{(m,n)}(x)$

m=1,n=1 $\varphi_1^{(1,1)}(x) = 1$

m=1,n=2 $\varphi_1^{(1,2)}(x) = \frac{x+3/2}{x+1}$

m=1,n=3 $\varphi_1^{(1,3)}(x) = \frac{x^2+5x/2+15/8}{(x+1)^2}$

m=2,n=2 $\varphi_1^{(2,2)}(x) = \frac{2x^2+4x+11/4}{(x+1)^2}$ and $\varphi_2^{(2,2)}(x) = -1$

m=2,n=3 $\varphi_1^{(2,3)}(x) = \frac{2x^3+7x^2+61x/8+57/16}{(x+1)^3}$ and $\varphi_2^{(2,3)}(x) = -\frac{x^2+5x+51/8}{(x+2)^2}$

m=3,n=3 $\varphi_1^{(3,3)}(x) = \frac{3x^4+12x^3+81x^2/4+117x/8+321/64}{(x+1)^4}$ and $\varphi_2^{(3,3)}(x) = -\frac{3x^4+24x^3+297x^2/4+855x/8+3993/64}{(x+2)^4}$ and $\varphi_3^{(3,3)}(x) = 1$