Analysis of the LS Estimation Error on a Rayleigh Fading Channel

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Abstract

A channel in a wireless communication link is often treated as time invariant over an estimation interval, during which a least squares estimate of the channel is calculated, using training symbols. By a second order Taylor expansion of the channel, the estimation error, due to time variation, can be approximated as a bias and an excess error, which are due to the curvature and linear change of the channel, respectively. Approximate expressions for the variance of the estimation error, in a Rayleigh fading channel, are presented here.

1. Introduction

To retrieve the transmitted signal in a mobile radio link, an estimate of the channel is needed. For a slowly time-varying channel this estimate can be obtained by using least squares (LS) over an estimation interval, during which the channel is treated as time invariant. Variations of the channel properties during the estimation interval will cause a channel estimation error. In this paper, this error is analyzed for the case of a Rayleigh fading channel.

A reasonable model for a mobile radio channel in the baseband, is a time-varying FIR filter [1]. The received data samples, $y[n]$ can thus be modeled as generated by a linear regression

$$y[n] = \boldsymbol{\varphi}^H[n]\mathbf{h}[n] + v[n], \tag{1}$$

where $\boldsymbol{\varphi}^H[n]$ is the known regressor vector consisting of M transmitted symbols $x[n]$

$$\boldsymbol{\varphi}[n] = [x[n]\ x[n-1] \dots\ x[n-M+1]]^H, \tag{2}$$

and where the unknown FIR channel is represented by a time-varying M-tap channel vector

$$\mathbf{h}[n] = [h_0[n]\ h_1[n] \dots h_{M-1}[n]]^T, \tag{3}$$

while $v[n]$ is a noise with zero mean and variance σ_v^2. Here, n is used to denote the index of a sample at symbol rate. Let

$$\mathbf{y} = [y[1]\ y[2] \dots\ y[N]]^T \tag{4}$$

$$\boldsymbol{\Phi} = [\boldsymbol{\varphi}[1]\ \boldsymbol{\varphi}[2] \dots \boldsymbol{\varphi}[N]]^H, \tag{5}$$

where $\mathbf{y}$ is a vector formed by the N previously received samples, and $\boldsymbol{\Phi}$ is a $N \times M$ matrix formed out of the $N + M - 1$ transmitted symbols. The LS estimate of the channel impulse response, $\hat{\mathbf{h}}_{\text{LS}}$, can then be found as [2]

$$\hat{\mathbf{h}}_{\text{LS}} = \boldsymbol{\Phi}^\dagger \mathbf{y} \triangleq \left(\boldsymbol{\Phi}^H \boldsymbol{\Phi}\right)^{-1} \boldsymbol{\Phi}^H \mathbf{y} \tag{6}$$

$$= \left(\sum_{n=1}^{N} \boldsymbol{\varphi}[n]\boldsymbol{\varphi}^H[n]\right)^{-1} \sum_{n=1}^{N} \boldsymbol{\varphi}[n]\boldsymbol{\varphi}^H[n]\mathbf{h}[n] + \boldsymbol{\Phi}^\dagger \mathbf{v}, \tag{7}$$

where $\mathbf{v}$ is a vector formed by the N previous noise terms, and $(\cdot)^\dagger$ denotes the Moore-Penrose pseudo-inverse. The channel estimate, $\hat{\mathbf{h}}_{\text{LS}}$, can thus be seen as a weighted average of $\mathbf{h}[n]$ over the estimation interval $[1\ N]$ corrupted by an additive noise. The desired channel is the true channel in the middle of the estimation interval. In the following, approximations for the variances of the different contributions to the channel estimation error will be derived.

2. Noise-Induced Error

From (7) the error due to measurement noise is found as

$$\mathbf{e}_{\text{N}} = \boldsymbol{\Phi}^\dagger \mathbf{v}. \tag{8}$$

For a white measurement noise independent of the training sequence, the covariance matrix for this error is given by [2]

$$\mathbf{P}_{\text{N}} = E_v \mathbf{e}_{\text{N}} \mathbf{e}_{\text{N}}^H = \sigma_v^2 (\boldsymbol{\Phi}^H \boldsymbol{\Phi})^{-1}, \tag{9}$$

where E_v denotes expectation over the noise. Assume $x[n]$ to be samples from a white complex circular sequence with zero mean and variance σ_x^2. Applying (46), from Appendix A, the ensemble average E_φ of $\mathbf{P}_{\text{N}}$ for different training sequences can be approximated as

$$\bar{\mathbf{P}}_{\text{N}} = \sigma_v^2 E_\varphi (\boldsymbol{\Phi}^H \boldsymbol{\Phi})^{-1} \approx \sigma_v^2 \frac{N + M + \kappa_x - 2}{N^2 \sigma_x^2} \mathbf{I}, \tag{10}$$

where κ_x is the Pearson kurtosis, which is defined as

$$\kappa_x \triangleq \frac{E|x[n]|^4}{(E|x[n]|^2)^2}. \quad (11)$$

For a complex Gaussian sequence $\kappa_x = 2$ and for a complex constant modulus sequence $\kappa_x = 1$. The sum of variances of all parameter estimation errors, given as the trace of $\bar{\mathbf{P}}_{\mathrm{N}}$, can by using (10) be approximated as

$$\operatorname{tr}\bar{\mathbf{P}}_{\mathrm{N}} \approx M\sigma_v^2 \frac{N+M+\kappa_x-2}{N^2\sigma_x^2}. \quad (12)$$

3. Excess Error

In the following we utilize a decomposition of the time varying channel into a sum of the average channel, $\bar{\mathbf{h}}_{[1\ N]}$, defined as

$$\bar{\mathbf{h}}_{[1\ N]} \triangleq \frac{1}{N}\sum_{n=1}^{N}\mathbf{h}[n], \quad (13)$$

and the time varying channel deviation from $\bar{\mathbf{h}}_{[1\ N]}$, $\boldsymbol{\vartheta}[n]$. Thus,

$$\mathbf{h}[n] = \bar{\mathbf{h}}_{[1\ N]} + \boldsymbol{\vartheta}[n]. \quad (14)$$

Using (14), the LS solution (7) can be expressed as

$$\hat{\mathbf{h}}_{\mathrm{LS}} = \bar{\mathbf{h}}_{[1\ N]} + \left(\mathbf{\Phi}^H\mathbf{\Phi}\right)^{-1}\sum_{n=1}^{N}\boldsymbol{\varphi}[n]\boldsymbol{\varphi}^H[n]\boldsymbol{\vartheta}[n] + \mathbf{e}_{\mathrm{N}}, \quad (15)$$

where the time invariant and the time varying terms are separated. To obtain expressions for the average channel and the deviation, we perform a second order Taylor expansion of the continuous channel, $\mathbf{h}(t)$, around the middle of the estimation interval

$$\begin{aligned}\mathbf{h}[n] = \mathbf{h}(nt_s) &\approx \mathbf{h}\left(\frac{N+1}{2}t_s\right)\\ &+ \frac{d\mathbf{h}}{dt}\left(\frac{N+1}{2}t_s\right)\left(n-\frac{N+1}{2}\right)t_s\\ &+ \frac{d^2\mathbf{h}}{dt^2}\left(\frac{N+1}{2}t_s\right)\left(n-\frac{N+1}{2}\right)^2\frac{t_s^2}{2}, \end{aligned} \quad (16)$$

where t_s denotes the sampling period. For oscillating channels, this Taylor expansion can be used for an estimation interval not longer than half a period of the fastest frequency component.

Using (16), the average channel can be expressed as

$$\begin{aligned}\bar{\mathbf{h}}_{[1\ N]} &= \frac{1}{N}\sum_{n=1}^{N}\mathbf{h}[n]\\ &\approx \mathbf{h}\left(\frac{N+1}{2}t_s\right) + \frac{d^2\mathbf{h}}{dt^2}\left(\frac{N+1}{2}t_s\right)\frac{N^2-1}{24}t_s^2. \end{aligned} \quad (17)$$

The deviation $\boldsymbol{\vartheta}[n]$ can be obtained from (16) and (17) as

$$\begin{aligned}\boldsymbol{\vartheta}[n] = \mathbf{h}[n] - \bar{\mathbf{h}}_{[1\ N]} &\approx \frac{d\mathbf{h}}{dt}\left(\frac{N+1}{2}t_s\right)\left(n-\frac{N+1}{2}\right)t_s\\ &+ \frac{d^2\mathbf{h}}{dt^2}\left(\frac{N+1}{2}t_s\right)\left[\left(n-\frac{N+1}{2}\right)^2 - \frac{N^2-1}{12}\right]\frac{t_s^2}{2}. \end{aligned} \quad (18)$$

Next we define the LS estimate's deviation from the average channel due to time variation, $\mathbf{e}_{\mathrm{E}}$, as the *excess error*

$$\mathbf{e}_{\mathrm{E}} = \hat{\mathbf{h}}_{\mathrm{LS}} - \bar{\mathbf{h}}_{[1\ N]} - \mathbf{e}_{\mathrm{N}}, \quad (19)$$

and note that $\mathbf{e}_{\mathrm{E}}$ is zero mean (as $\mathbf{h}(t)$ and it's derivatives are zero mean). The covariance matrix for the excess error, $\mathbf{P}_{\mathrm{E}}$, is found by averaging, E_h, over channel realizations as

$$\begin{aligned}\mathbf{P}_{\mathrm{E}} &= E_h\mathbf{e}_{\mathrm{E}}\mathbf{e}_{\mathrm{E}}^H \quad (20)\\ &= E_h\mathbf{Q}\sum_{n=1}^{N}\sum_{m=1}^{N}\left(\boldsymbol{\varphi}[n]\boldsymbol{\varphi}^H[n]\boldsymbol{\vartheta}[n]\boldsymbol{\vartheta}^H[m]\boldsymbol{\varphi}[m]\boldsymbol{\varphi}^H[m]\right)\mathbf{Q},\end{aligned}$$

where $(\mathbf{\Phi}^H\mathbf{\Phi})^{-1} \triangleq \mathbf{Q}$. To compute $\mathbf{P}_{\mathrm{E}}$ using the second-order Taylor approximation in (16), we need to first compute the covariance and cross-covariance matrices of the first and second derivatives of the channel. For a *Rayleigh fading* channel these are obtained as (see Appendix B)

$$E_h\mathbf{h}(t)\mathbf{h}^H(t) = \mathbf{R}_h \quad (21)$$

$$E_h\frac{d\mathbf{h}}{dt}\frac{d\mathbf{h}^H}{dt} = \frac{\omega_d^2}{2}\mathbf{R}_h \quad (22)$$

$$E_h\frac{d\mathbf{h}}{dt}\frac{d^2\mathbf{h}^H}{dt^2} = 0 \quad (23)$$

$$E_h\frac{d^2\mathbf{h}}{dt^2}\frac{d^2\mathbf{h}^H}{dt^2} = \frac{3\omega_d^4}{8}\mathbf{R}_h \quad (24)$$

where ω_d is the Doppler frequency. For moderately fast fading, $\omega_d t_s \ll 1$. The contribution from the second derivative of the channel in (18) can then be neglected and only the inclination is considered. Using (22) and the first term of (18), the covariance matrix of the excess error (20) can be approximated as

$$\mathbf{P}_{\mathrm{E}} \approx \mathbf{Q}\sum_{n=1}^{N}\sum_{m=1}^{N} f[n,m]\boldsymbol{\varphi}[n]\boldsymbol{\varphi}^H[n]\mathbf{R}_h\boldsymbol{\varphi}[m]\boldsymbol{\varphi}^H[m]\mathbf{Q}, \quad (25)$$

where the scalar function $f[n,m]$ is defined as

$$f[n,m] \triangleq \frac{(t_s\omega_d)^2}{2}\left(n-\frac{N+1}{2}\right)\left(m-\frac{N+1}{2}\right). \quad (26)$$

Thus, for any given training sequence, the covariance matrix of the LS excess error due to time variation of a Rayleigh fading channel can be calculated using (25) and (26).

Given the distribution of the training sequences, an approximate average covariance matrix $\bar{\mathbf{P}}_{\rm E}$ for an ensemble of training sequences can be calculated. Let the matrix part of the sum in (25) be denoted

$$\Xi[n,m] = \varphi[n]\varphi^H[n]\mathbf{R}_h\varphi[m]\varphi^H[m] \tag{27}$$

Note that

$$\varphi^H[n]\mathbf{R}_h\varphi[m] = \sum_{k=1}^{M}\sum_{l=1}^{M} x[n-k+1]x^*[m-l+1]r_{k,l}, \tag{28}$$

where $r_{k,l} = \mathbf{R}_h^{k,l}$ is the k,l:th element of the channel covariance matrix, is scalar and can thus be moved to the end of the product. The average E_φ over training sequences of the i,j:th element of $\Xi[n,m]$ is [3]

$$\begin{aligned} &E_\varphi \Xi^{i,j}[n,m] = \\ &E_\varphi \sum_{k=1}^{M}\sum_{l=1}^{M} r_{k,l}x^*[n-i+1]x[m-j+1]x[n-k+1]x^*[m-l+1] \\ &\quad = \sigma_x^4 \sum_{k=1}^{M}\sum_{l=1}^{M} r_{k,l}\left(\delta_{i,k}\delta_{j,l} + \delta_{n-i,m-j}\delta_{n-k,m-l}\right. \\ &\qquad \left. + (\kappa_x-2)\delta_{i,k}\delta_{j,l}\delta_{n-i,m-j}\right), \end{aligned} \tag{29}$$

To find the variance of the excess error on the individual taps we need only to calculate the diagonal elements of the covariance matrix $\bar{\mathbf{P}}_{\rm E}$ and thus only $E_\varphi \Xi^{i,j}[n,m]$ for $i=j$, which according to (29) renders

$$\begin{aligned} &E_\varphi \Xi^{i,i}[n,m] = \\ &= \sigma_x^4\left(r_{i,i} + \delta_{n,m}\sum_{k=1}^{M}\sum_{l=1}^{M} r_{k,l}\delta_{n-k,m-l} + \delta_{n,m}(\kappa_x-2)r_{i,i}\right) \\ &= \begin{cases} \sigma_x^4\left(r_{i,i} + \sum_{k=1}^{M} r_{k,k} + (\kappa_x-2)r_{i,i}\right) & n=m \\ \sigma_x^4 r_{i,i} & n \neq m \end{cases} \end{aligned} \tag{30}$$

The expectation E_φ of the summation over n,m in (25) can now be calculated as

$$\begin{aligned} &E_\varphi \sum_{n=1}^{N}\sum_{m=1}^{N} f[n,m]\Xi^{i,i}[n,m] = \\ &\sigma_x^4 \frac{(t_s\omega_d)^2}{2}\frac{N(N^2-1)}{12}\left(\sum_{k=1}^{M}\sigma_{h_k}^2 + (\kappa_x-2)\sigma_{h_i}^2\right). \end{aligned} \tag{31}$$

where $\sigma_{h_k}^2 = r_{k,k}$ is the variance of tap k. The diagonal elements of $\bar{\mathbf{P}}_{\rm E}$, denoted by $\sigma_{{\rm E}_i}^2$, can by inserting (30) in (25), be approximated as

$$\begin{aligned} \sigma_{{\rm E}_i}^2 &= E_\varphi \mathbf{Q}\sum_{n=1}^{N}\sum_{m=1}^{N} f[n,m]\Xi^{i,i}[n,m]\mathbf{Q} \\ &\approx \frac{(t_s\omega_d)^2}{24}(N + 2(M+\kappa_x-2))\left(\sum_{k=1}^{M}\sigma_{h_k}^2 + (\kappa_x-2)\sigma_{h_i}^2\right), \end{aligned} \tag{32}$$

where all factors have been treated as independent and the first and last factors of (25) are approximated as their ensemble averages, as in (46) in Appendix A.

To express the covariance as a function of the time-frequency product, the Doppler frequency ω_d has to be expressed as $\omega_d = 2\pi f_d$. The sum of variances of the excess errors for all parameters, given as the trace of $\bar{\mathbf{P}}_{\rm E}$, can then, using (32), be approximated as

$$\begin{aligned} &\operatorname{tr}\bar{\mathbf{P}}_{\rm E} = \sum_{k=1}^{M}\sigma_{{\rm E}_k}^2 \approx \\ &\frac{\pi^2(Tf_d)^2}{6N}\left(1+\frac{2(M+\kappa_x-2)}{N}\right)(M+\kappa_x-2)\sum_{k=1}^{M}\sigma_{h_k}^2, \end{aligned} \tag{33}$$

where $T = Nt_s$ denotes the length of the estimation interval.

4. Bias Error

The LS estimate (7) deviates from the value of the channel at the middle of the interval. Insert (17) in (19),

$$\begin{aligned} \hat{\mathbf{h}}_{\rm LS} &= \hat{\mathbf{h}}\left(\frac{N+1}{2}t_s\right) = \bar{\mathbf{h}}_{[1\ N]} + \mathbf{e}_{\rm E} + \mathbf{e}_{\rm N} \\ &\approx \mathbf{h}\left(\frac{N+1}{2}t_s\right) + \frac{d^2\mathbf{h}}{dt^2}\left(\frac{N+1}{2}t_s\right)\frac{N^2-1}{24}t_s^2 + \mathbf{e}_{\rm E} + \mathbf{e}_{\rm N}. \end{aligned} \tag{34}$$

The second term in (34) can be viewed as a *bias error* [4],

$$\mathbf{e}_{\rm B} \approx \frac{d^2\mathbf{h}}{dt^2}\left(\frac{N+1}{2}t_s\right)\frac{N^2-1}{24}t_s^2, \tag{35}$$

which depends on the curvature of the channel in the estimation interval. Averaging over realizations of the channel, the covariance matrix for $\mathbf{e}_{\rm B}$ is

$$\begin{aligned} \mathbf{P}_{\rm B} = E_h \mathbf{e}_{\rm B}\mathbf{e}_{\rm B}^H &\approx \left(\frac{N^2-1}{24}t_s^2\right)^2 E_h \frac{d^2\mathbf{h}}{dt^2}\frac{d^2\mathbf{h}}{dt^2}^H \\ &= \frac{(N^2-1)^2}{1536}(t_s\omega_d)^4\mathbf{R}_h \approx \frac{\pi^4(Tf_d)^4}{96}\mathbf{R}_h. \end{aligned} \tag{36}$$

The variance of the bias error does thus solely depend on the length of the estimation interval and not on the statistics of the training sequence. The trace of $\mathbf{P}_{\rm B}$ can be approximated as

$$\operatorname{tr}\mathbf{P}_{\rm B} \approx \frac{\pi^4(Tf_d)^4}{96}\sum_{k=1}^{M}\sigma_{h_k}^2. \tag{37}$$

Note that the Taylor expansion (16) is valid for no more than half a period of the fastest oscillation. Thus, the approximations are only valid for $Tf_d < 1/2$.

5. Total Estimation Error

The LS estimate of the impulse response, at the middle of the estimation interval, can be written as

$$\hat{\mathbf{h}}_{\mathrm{LS}} = \mathbf{h}\left(\frac{N+1}{2}t_s\right) + \mathbf{e}_{\mathrm{LS}} \tag{38}$$

where the additive error term is approximated as

$$\mathbf{e}_{\mathrm{LS}} \approx \mathbf{e}_{\mathrm{N}} + \mathbf{e}_{\mathrm{E}} + \mathbf{e}_{\mathrm{B}}, \tag{39}$$

where the error terms are given as in (8), (19) and (35). According to (23), $\mathbf{e}_{\mathrm{E}}$ and $\mathbf{e}_{\mathrm{B}}$ are approximately uncorrelated, since $\mathbf{e}_{\mathrm{E}}$ is related to $d\mathbf{h}/dt$ and $\mathbf{e}_{\mathrm{B}}$ to $d\mathbf{h}^2/d^2t$. Furthermore, under the assumption that the measurement noise is independent of the channel, $\mathbf{e}_{\mathrm{N}}$ is uncorrelated to both $\mathbf{e}_{\mathrm{E}}$ and $\mathbf{e}_{\mathrm{B}}$. The covariance matrix of the additive noise can thus be modeled as

$$\mathbf{P}_{\mathrm{LS}} \approx \bar{\mathbf{P}}_{\mathrm{N}} + \bar{\mathbf{P}}_{\mathrm{E}} + \mathbf{P}_{\mathrm{B}}. \tag{40}$$

The sum of parameter error variance can be obtained as the trace of $\mathbf{P}_{\mathrm{LS}}$, as obtained from summing (12), (33) and (37)

$$\mathrm{tr}\,\mathbf{P}_{\mathrm{LS}} \approx \frac{\pi^2(Tf_d)^2}{6N}\frac{N+2(M+\kappa_x-2)}{N}(M+\kappa_x-2)\sum_{k=1}^{M}\sigma_{h_k}^2 + \frac{\pi^4(Tf_d)^4}{96}\sum_{k=1}^{M}\sigma_{h_k}^2 + M\sigma_v^2\frac{N+M+\kappa_x-2}{N^2\sigma_x^2}. \tag{41}$$

6. Simulation

In our simulation example, we let each tap in an approximate Rayleigh fading channel be simulated as the sum of 500 complex sinusoids with Gaussian distributed amplitude and frequency $f_d\cos(\theta_n)$, where θ_n is uniformly distributed between $[0\ 2\pi[$. The channel has four taps with exponentially decaying variances 1, 1/2, 1/4 and 1/8, respectively. The Doppler frequency is $f_d = 160$ Hz and the sampling period is $t_s = 5\ \mu$s. A white QPSK signal is transmitted over the time-varying channel and 10^5 samples are measured by the receiver. A measurement noise is added so that the received SNR is 20 dB.

The channel is estimated block-wise using a LS estimator using different numbers of training symbols. The estimated channel is compared to the true channel and the error is calculated for the estimated taps. The variance of the error is estimated and summed for all the taps in the channel to obtain $\mathrm{tr}\,\hat{\mathbf{P}}_{\mathrm{LS}}$. The theoretical value for $\mathrm{tr}\,\mathbf{P}_{\mathrm{LS}}$ is obtained from (41). Both the estimated and the theoretical approximation of $\mathrm{tr}\,\mathbf{P}_{\mathrm{LS}}$ are divided by $\sum_{k=1}^{M}\sigma_{h_k}^2$ to obtain the normalized means square error (NMSE). As seen in Figure 1, the theoretical approximation almost coincides with the result for the NMSE from the simulation for time-frequency products, Tf_d, below $1/4$ (344 samples) and after that the NMSE is slightly overestimated.

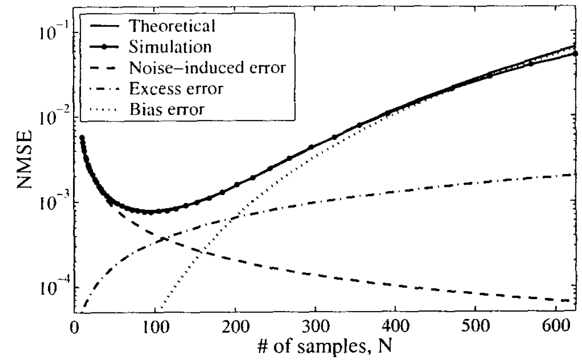

Figure 1. The variance of estimation error for the simulated channels is compared with the theoretical values obtained from (41). The contributions from (12), (33) and (37) are plotted separately. All variances are normalized by the total power of the channel.

Appendix A

In the expressions for the covariance matrices of the noise-induced error, (9), and the excess error, (20), the inverse of the sample covariance matrix plays an important role. For a white training sequence, $x[n]$, the matrix product $\mathbf{\Phi}^H\mathbf{\Phi}$ can be decomposed as

$$\mathbf{\Phi}^H\mathbf{\Phi} = \sum_{n=1}^{N}\varphi[n]\varphi^H[n] = N\mathbf{R}_x + \sum_{n=1}^{N}\mathbf{Z}[n]. \tag{42}$$

where $\mathbf{R}_x = \sigma_x^2\mathbf{I}$ is the covariance matrix for $x[n]$, and

$$\mathbf{Z}[n] = \varphi[n]\varphi^H[n] - \mathbf{R}_x \tag{43}$$

is the zero mean deviation from this covariance matrix. For a circular complex valued sequence, it holds [3]

$$E_{\varphi}\mathbf{Z}[n]\mathbf{Z}[m] = \sigma_x^4(M+\kappa_x-2)\mathbf{I}\delta_{n,m}. \tag{44}$$

To obtain an estimate of the inverse sample covariance matrix, we make a second order Taylor expansion around $\mathbf{I}$ as

$$\mathbf{Q} \triangleq \left(\mathbf{\Phi}^H\mathbf{\Phi}\right)^{-1} = \frac{1}{N\sigma_x^2}\left(\mathbf{I} + \frac{1}{N\sigma_x^2}\sum_{n=1}^{N}\mathbf{Z}[n]\right)^{-1}$$
$$\approx \frac{1}{N\sigma_x^2}\left(\mathbf{I} - \frac{1}{N\sigma_x^2}\sum_{n=1}^{N}\mathbf{Z}[n] + \frac{1}{N^2\sigma_x^4}\sum_{n=1}^{N}\sum_{m=1}^{N}\mathbf{Z}[n]\mathbf{Z}[m]\right), \tag{45}$$

The expected value of (45), using (44), yields

$$E_{\varphi}\left(\mathbf{\Phi}^H\mathbf{\Phi}\right)^{-1} \approx \frac{1}{N\sigma_x^2}\left(\mathbf{I} + \frac{M+\kappa_x-2}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N}\mathbf{I}\delta_{n,m}\right) = \frac{N+M+\kappa_x-2}{N^2\sigma_x^2}\mathbf{I}. \quad (46)$$

Appendix B

To compute $\mathbf{P}_{\rm E}$ in (20) and $\mathbf{P}_{\rm B}$ in (36), the covariance and cross-covariance matrices of the first and second derivatives of the channel are needed. For a Rayleigh fading channel the autocorrelation matrix is given by [5]

$$E_h\mathbf{h}(t)\mathbf{h}^H(t+\tau) = \mathbf{R}_h J_0(\omega_d\tau), \quad (47)$$

where $J_0(\cdot)$ denotes the zero order Bessel function of the first kind [6],

$$J_0(\omega_d\tau) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{\omega_d\tau}{2}\right)^{2k}, \quad (48)$$

and ω_d is the Doppler frequency (in rad/s). Using (47) and (48) the covariance matrices of the derivatives of the channel can be derived. The covariance matrix of the first derivative is

$$E_h\frac{d\mathbf{h}}{dt}\frac{d\mathbf{h}^H}{dt} = E_h\lim_{\Delta t,\Delta\tau\to 0,0}\frac{\mathbf{h}(t+\Delta t)-\mathbf{h}(t)}{\Delta t}\frac{\mathbf{h}^H(t+\Delta\tau)-\mathbf{h}^H(t)}{\Delta\tau}$$
$$= \lim_{\Delta t,\Delta\tau\to 0,0}\frac{\mathbf{R}_h}{\Delta t\Delta\tau}(J_0(\omega_d(\Delta t-\Delta\tau)) - J_0(\omega_d\Delta t) - J_0(\omega_d\Delta\tau)+1). \quad (49)$$

Only orders up to two in the series expansion of $J_0(\cdot)$ are needed as higher orders will cancel or have terms that will approach zero in the limit.

$$E_h\frac{d\mathbf{h}}{dt}\frac{d\mathbf{h}^H}{dt} = \lim_{\Delta t,\Delta\tau\to 0,0}\frac{\mathbf{R}_h}{\Delta t\Delta\tau}\left(1-\frac{\omega_d^2(\Delta t-\Delta\tau)^2}{4} -1+\frac{\omega_d^2\Delta t^2}{4}-1+\frac{\omega_d^2\Delta\tau^2}{4}+1\right) = \frac{\omega_d^2}{2}\mathbf{R}_h. \quad (50)$$

The cross-covariance between the first and second derivative of the channel is

$$E_h\frac{d\mathbf{h}}{dt}\frac{d^2\mathbf{h}^H}{dt^2} = E_h\lim_{\Delta t,\Delta\tau\to 0,0}\frac{\mathbf{h}(t+\Delta t)-\mathbf{h}(t-\Delta t)}{2\Delta t}\times\frac{\mathbf{h}^H(t+\Delta\tau)-2\mathbf{h}^H(t)+\mathbf{h}^H(t-\Delta\tau)}{\Delta\tau^2}$$
$$= \lim_{\Delta t,\Delta\tau\to 0,0}\frac{\mathbf{R}_h}{\Delta t\Delta\tau^2}(J_0(\omega_d(\Delta t-\Delta\tau)) - 2J_0(\omega_d\Delta t) + J_0(\omega_d(\Delta t+\Delta\tau)) - J_0(\omega_d(\Delta t-\Delta\tau)) + 2J_0(\omega_d\Delta t) - J_0(\omega_d(\Delta t+\Delta\tau))) = 0. \quad (51)$$

The covariance matrix for the second derivative is

$$E_h\frac{d^2\mathbf{h}}{dt^2}\frac{d^2\mathbf{h}^H}{dt^2} = E_h\lim_{\Delta t,\Delta\tau\to 0.0}\frac{\mathbf{h}^H(t+\Delta t)-2\mathbf{h}^H(t)+\mathbf{h}^H(t-\Delta t)}{\Delta t^2}\times\frac{\mathbf{h}^H(t+\Delta\tau)-2\mathbf{h}^H(t)+\mathbf{h}^H(t-\Delta\tau)}{\Delta\tau^2}$$
$$= \lim_{\Delta t,\Delta\tau\to 0,0}\frac{\mathbf{R}_h}{\Delta t^2\Delta\tau^2}(4+2J_0(\omega_d(\Delta t+\Delta\tau)) + 2J_0(\omega_d(\Delta t-\Delta\tau)) - 4J_0(\omega_d\Delta t) - 4J_0(\omega_d\Delta\tau))). \quad (52)$$

Series expansion of $J_0(\cdot)$ up to the forth order is needed, higher order terms cancel or approach zero in the limit.

$$E_h\frac{d^2\mathbf{h}}{dt^2}\frac{d^2\mathbf{h}^H}{dt^2} = \lim_{\Delta t,\Delta\tau\to 0,0}\frac{\mathbf{R}_h}{\Delta t^2\Delta\tau^2}\Bigg(4 + 2\left(1-\frac{\omega_d^2(\Delta t+\Delta\tau)^2}{4}+\frac{\omega_d^4(\Delta t+\Delta\tau)^4}{64}\right) + 2\left(1-\frac{\omega_d^2(\Delta t-\Delta\tau)^2}{4}+\frac{\omega_d^4(\Delta t-\Delta\tau)^4}{64}\right) - 4\left(2-\frac{\omega_d^2(\Delta t^2+\Delta\tau^2)}{4}+\frac{\omega_d^4(\Delta t^4+\Delta\tau^4)}{64}\right)\Bigg) = \frac{3\omega_d^4}{8}\mathbf{R}_h \quad (53)$$

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